Lesson 25

Quadrilaterals and Polygons

Aims

The aim of this lesson is to enable you to:

- classify and define various types of quadrilaterals
- calculate angles in quadrilaterals
- recognise symmetries and name polygons
- calculate exterior and interior angles
- know and use angle and symmetry properties of quadrilaterals and other polygons

Context

Quadrilaterals and polygons play a role in many constructions, for example tiling patterns and plans for buildings. The symmetries of the shapes are often useful in such cases.

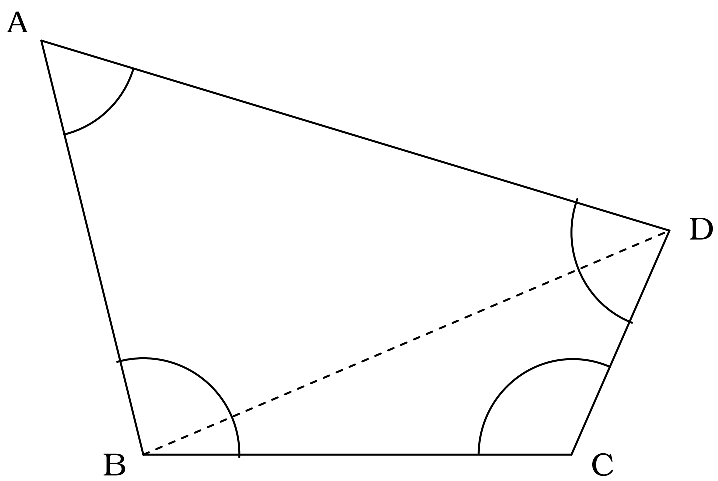
Oxford Open Learning

Quadrilaterals

A **quadrilateral** is any shape with four sides. However, the sides must be straight lines: curves are not allowed.

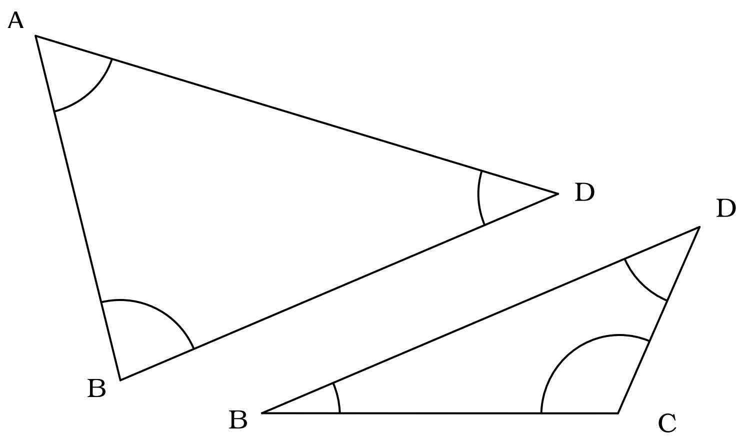
It is a fact that the four angles of any quadrilateral add up to 360°. One way to verify this fact would be to draw lots of quadrilaterals of different sizes and shapes, and then add up the angles in each.

However, there is another way of establishing this important fact. This technique is itself important, because it is the basis of a more general method that is used later in this lesson. Let ABCD be any quadrilateral. Choose any vertex, and join this up with the opposite vertex. In the diagram, vertex B was chosen, and then joined up with D.



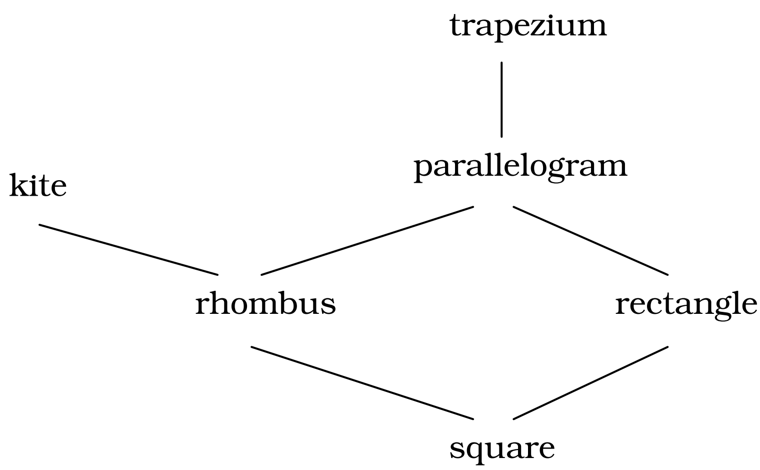
The four angles of the quadrilateral are marked. We require the total of these four angles. If you look carefully, you will hopefully see that the total of these four angles is the same as the total of the three angles of triangle ABD and the three angles of triangle BCD. To help you see this, the quadrilateral is cut along the dotted line BD to make the triangles ABD and BCD. These two triangles are shown separately in the following diagram.

We know that the three angles of any triangle add up to 180°, so that the angles of two triangles add up to 360°. We can therefore conclude that the four angles of any quadrilateral add up to 360°.



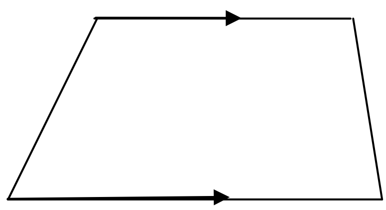
Types of Quadrilateral

The following diagram shows the relationships between the different members of the family of quadrilaterals. The shapes lower down are more specialised than those above.



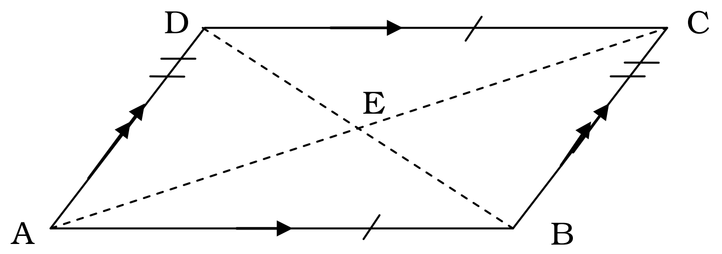
Trapezium

The trapezium has one pair of parallel sides. In general, a trapezium has no line of symmetry (although this is possible). The trapezium has no rotational symmetry.



Parallelogram

The parallelogram has two pairs of parallel sides. Each side in a parallelogram is parallel to the side opposite: AB is parallel to DC and AD is parallel to BC. This is the defining property of a parallelogram.



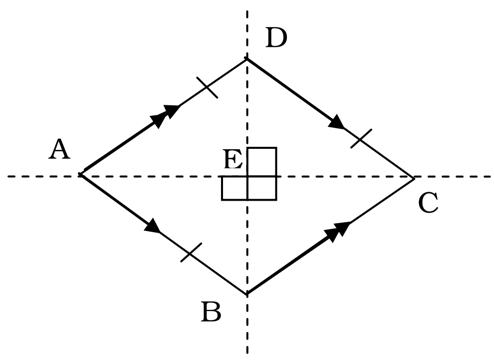
However, there are many other properties that you need to know:

1. Opposite angles are equal: angle A = angle C, and angle B = angle D.
2. Adjacent angles add up to 180°: angles A and B add up to 180°, angles A and D add up to 180° etc.
3. The diagonals bisect each other: E is the midpoint of both the diagonals AC and BD.
4. There is rotational symmetry of order 2: a half-turn about E makes the parallelogram coincide with itself.

Rhombus

A rhombus is a special parallelogram that has four equal sides. So the defining properties of a rhombus are that is has four equal sides and opposite pairs of sides are parallel.

In the diagram below, AB = BC = CD = DA. Also, AB is parallel to CD, and BC is parallel to AD.



Since a rhombus is a special parallelogram, it has all the properties of a parallelogram:

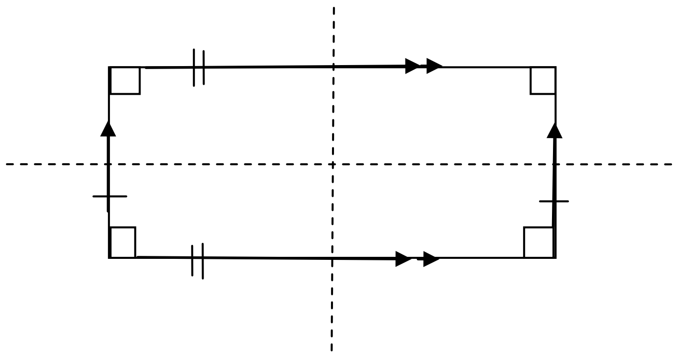
1. Opposite angles are equal.
2. Adjacent angles add up to 180°.
3. The diagonals bisect each other.
4. There is rotational symmetry of order 2.

However, a rhombus has some additional properties:

5. Two lines of symmetry (shown as dashed lines in the diagram).
6. The diagonals (which are also the lines of symmetry) not only bisect each other, but do so at right-angles.

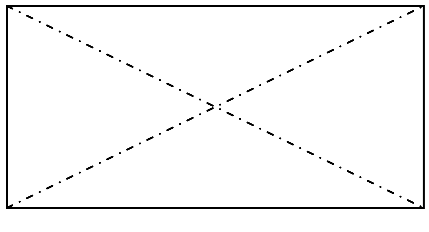
Rectangle

Rectangles are familiar, but we still need to look at them in detail. A rectangle is a special parallelogram, where all the angles are equal. Since the angles add up to 360°, this means that each angle is a right-angle.



The properties of a rectangle are as follows:

1. Opposite sides are parallel and equal in length.
2. All angles are right-angles.
3. There are two lines of symmetry: shown as dashed lines in the diagram.
4. There is rotational symmetry of order 2.
5. The diagonals (shown dashed in the diagram below) bisect each other, but not at right-angles.

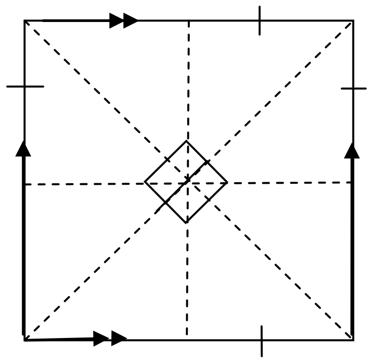


Square

The square is also familiar. But again we need to consider all these new aspects of quadrilaterals.

A square can be thought of as a special rectangle, where all the sides are of equal length. A square can also be thought of as a special rhombus, where all the angles are equal. Whichever way we think of it, a square has the following properties:

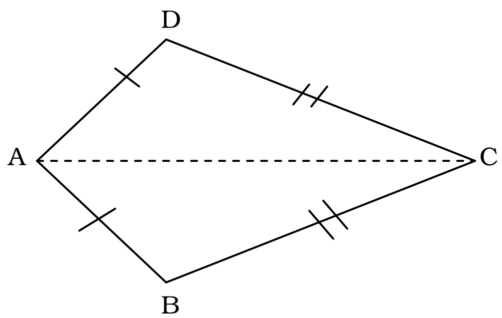
1. All four sides are equal.
2. All four angles are right-angles.
3. Four lines of symmetry (shown dashed in the diagram).
4. Rotational symmetry of order 4. A rotation of a quarter-turn will cause a square to coincide with itself.
5. The diagonals bisect each other at right-angles.



Kite

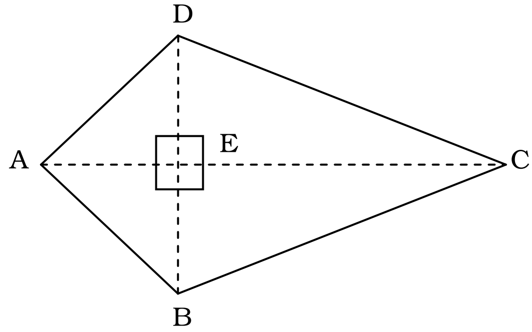
A kite is a quadrilateral with two pairs of sides which are equal in length. However, unlike a rectangle or a parallelogram, the equal sides are not opposite to each other, they are next to each other.

The quadrilateral ABCD is a kite with the line of symmetry AC. The sides AB and AD are equal in length. Similarly, the sides BC and CD are equal in length.



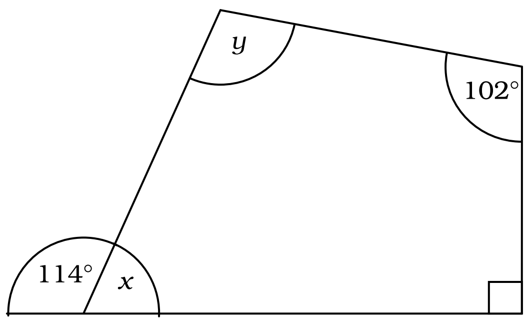
The additional properties of the kite are as follows:

1. One line of symmetry (shown dashed).
2. One pair of equal angles: at B and D in the diagram, since AC is a mirror line.
3. No rotational symmetry: the only rotation to make the shape coincide with itself is a full turn of 360°.
4. The diagonals cut each other at right-angles. Note that the diagonal AC bisects the diagonal BD, but that BD does not bisect AC. In other words, E is the mid-point of BD but not of AC.



Example 1

Find the missing angles denoted by *x* and *y* in the diagram:



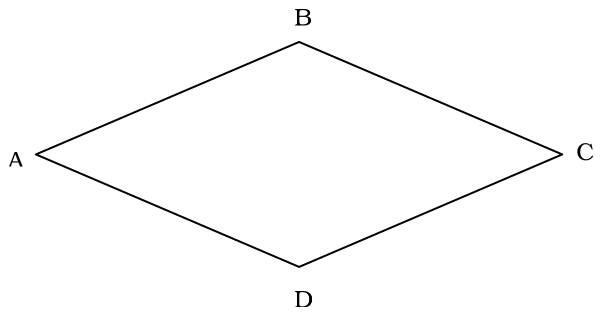
The angles on a straight line add up to 180°, so $114 + x = 180$. x must therefore be 66°.

The four angles of the quadrilateral add up to 360°, so $x + y + 102 + 90 = 360$. We know x is 66°, so:

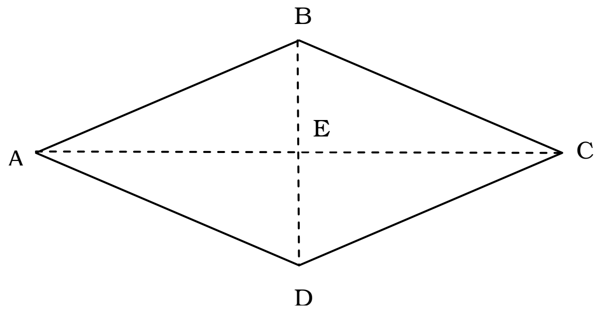
$66 + y + 102 + 90 = 360$
$y + 258 = 360$
$y = 360 - 258$
$y = 102°$.

Example 2

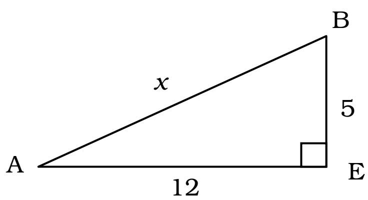
In the rhombus ABCD, AC = 24 cm and BD = 10 cm. Find the lengths of the sides of the rhombus.



Draw the diagonals of the rhombus: AC and BD.



Now we use the fact that the diagonals of a rhombus bisect each other at right angles (at the point E). This means that the triangle ABE is right-angled. Furthermore, AE is half the length of AC, so that AE is 12 cm, and BE is half the length of BD, so that BE is 5 cm.



Now we can use Pythagoras' Theorem in triangle ABE:

$$x^2 = 12^2 + 5^2$$
$$x^2 = 144 + 25$$
$$x^2 = 169$$
$$x = \sqrt{169}$$
$$x = 13$$

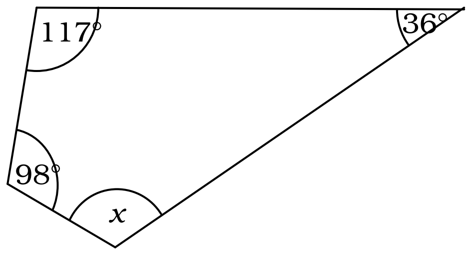
So AB is 13 cm long. But the sides of a rhombus are all equal, so that all the sides of ABCD are 13 cm long.

Activity 1

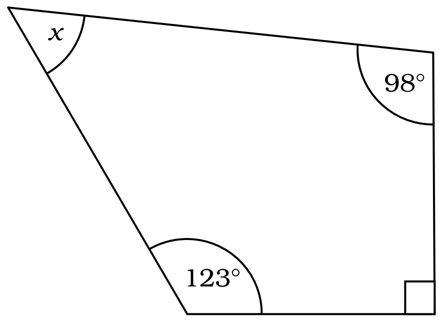
Now tackle the following exercises.

Find the missing angles denoted by letters:

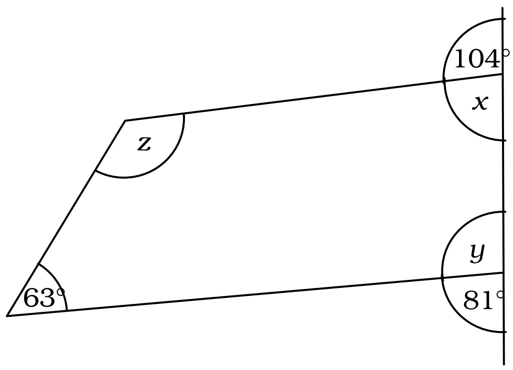
1.



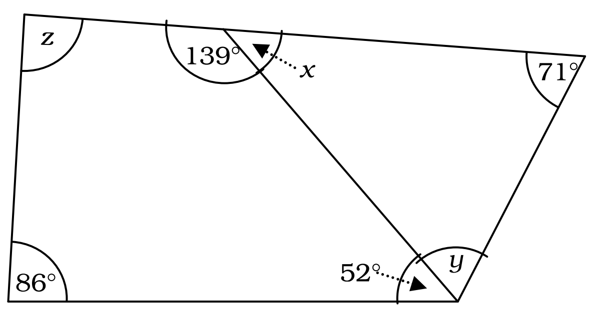
2.



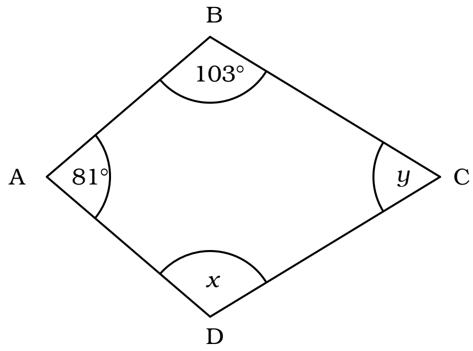
3.



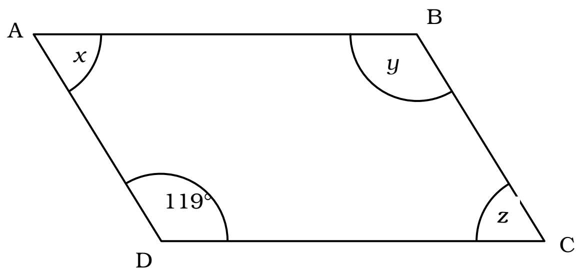
4.



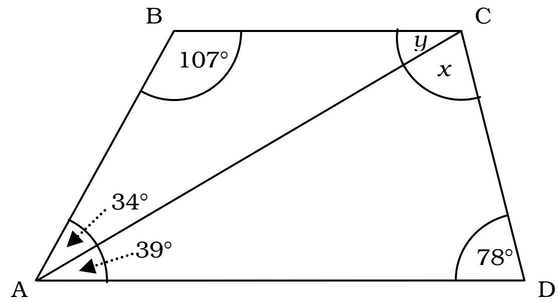
5. ABCD is a kite, with AC as the axis of symmetry.



6. ABCD is a parallelogram.



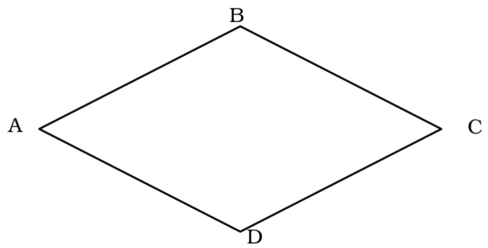
7.(a) Find the angles denoted by *x* and *y*.
(b)Check that the angles in quadrilateral ABCD sum to 360°.



8 A square has each side 3.4 m long. Use Pythagoras' Theorem (and a calculator) to find the length of a diagonal of this square correct to three significant figures.

9. A rectangle has to sides with lengths 28 cm and 45 cm. Use Pythagoras' Theorem (and a calculator) to find the length of a diagonal of this rectangle.

10. The rhombus ABCD is such that AC = 30 cm and BD = 16 cm. Find the length of each side of the rhombus.



Activity 2

Draw and name the quadrilateral that has the properties specified below. Use this as a checklist. You need to know all these names and properties very well.

1. one pair of parallel sides

2. opposite sides parallel
   all sides equal
   all angles right-angles

3. opposite sides parallel and equal
   all angles right-angles

4. opposite sides parallel and equal
   opposite angles equal

5. opposite sides parallel
   all sides equal
   opposite angles equal

6. two adjacent sides equal
   the other two adjacent sides equal

Polygons

A **polygon** just means a shape that has straight sides. A triangle is a polygon with three sides. A quadrilateral is a polygon with four sides. The following table shows the names of some more polygons. It is unlikely that you need to know the names of any of these apart from pentagon, hexagon and octagon.

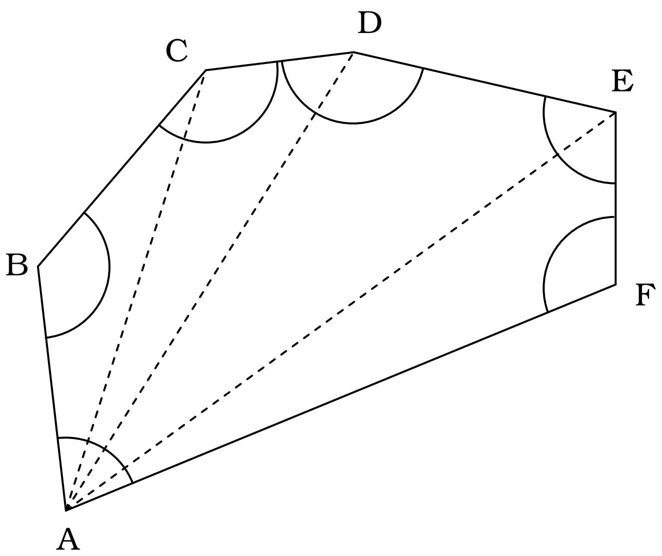
Name	**Number of sides**
Pentagon	5
Hexagon	6
Heptagon	7
Octagon	8
Nonagon	9
Decagon	10
Hendecagon	11
Dodecagon	12

Sum of the Interior Angles of a Polygon

There is a method for working out the sum of the interior angles of any polygon. Choose one vertex, and then draw lines to all other possible vertices to make a number of triangles.

For example, in the diagram below, the original shape is a hexagon (six sides). Vertex A is chosen, and three dashed lines are drawn: AC, AD and AE. There are now four triangles: ABC, ACD, ADE and AEF. Look carefully at the diagram. We want to know the sum of the six angles of the original hexagon.

This sum is the same as the total of all twelve angles of the four new triangles. But we know that the angles of any triangle add up to 180°. The angles of four triangles therefore add up to 720°. We can conclude that the six interior angles of the original hexagon add up to 720°. There is nothing special about this hexagon: we have in fact proved that the six interior angles of **any** hexagon add up to 720°.



The above technique can be adapted for any polygon. Note that for the hexagon, the number of triangles is two less than the number of sides of the hexagon. This relationship is always true. If the original shape had, say, ten sides, then we would have drawn seven lines to make eight triangles. The number of triangles is always two less than the number of sides of the original shape. Try it and see.

If a polygon has n sides, then there will be $n - 2$ triangles. The angles in each triangle add up to 180°, so the angles in a polygon with n sides add up to $(n-2)\times 180°$. This formula is sometimes quoted in the alternative form: $(2n - 4)$ right-angles.

Example 1

Find the sum of the interior angles of any pentagon.

A pentagon has five sides, so $n = 5$. Using the formula $(n-2)\times 180$, the angle sum is $(5-2)\times 180° = 3\times 180° = 540°$.

Sum of the Exterior Angles of a Polygon

As the name implies, exterior angles are outside the polygon. In the following diagram, the exterior angles of the pentagon ABCDE are the angles marked *a, b, c, d* and *e*.

Imagine you are going to walk around the pentagon. Let us be precise. You are going to walk anticlockwise around the pentagon. The starting and finishing point will be A, and the starting and finishing direction must be along AB.

Ready? You are at A and facing towards B. Walk along AB until you reach B. Now you need to turn anticlockwise through the angle *b* so that you are ready to walk along BC. Now walk along BC until you arrive at C. You now need to turn anticlockwise through the angle *c* so that you are ready to walk along CD. Now walk along CD until you arrive at D. You now need to turn anticlockwise through the angle *d* so that you are ready to walk along DE.

Now walk along DE until you arrive at E. You now need to turn anticlockwise through the angle *e* so that you are ready to walk along EA. Now walk along EA until you get back to point A. However, you need to finish exactly as you started, facing along AB, so you now need to turn anticlockwise through the angle *a*.

Now think about the overall effect of your journey. You have started at A and returned to A. The starting direction is the same as the direction at the finish. You must therefore have turned through 360° anticlockwise. We have shown that

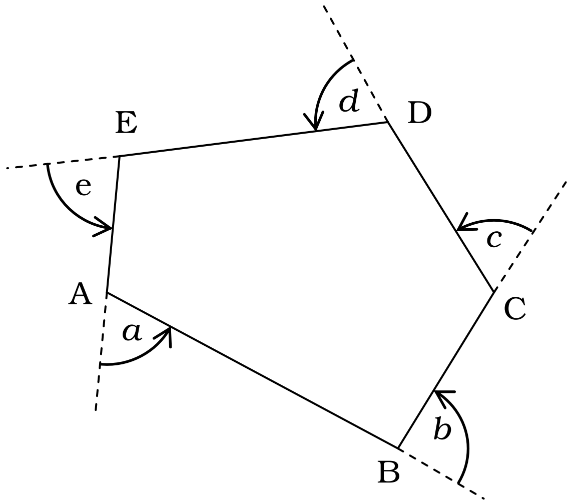
$$a + b + c + d + e = 360°$$

in other words, the exterior angles of the pentagon add up to 360°.

Now, there is very little special about the pentagon. There is, however, one point to note. The pentagon is **convex**: none of the interior angles are **reflex**. So we have shown that the exterior angles of any convex pentagon add up to 360°.

There is also nothing special about the number of sides. It does not matter whether we walk round a (convex) polygon with 3 or 103 sides: we always turn through 360°. We have therefore established that:

The exterior angles of any convex polygon add up to 360°.

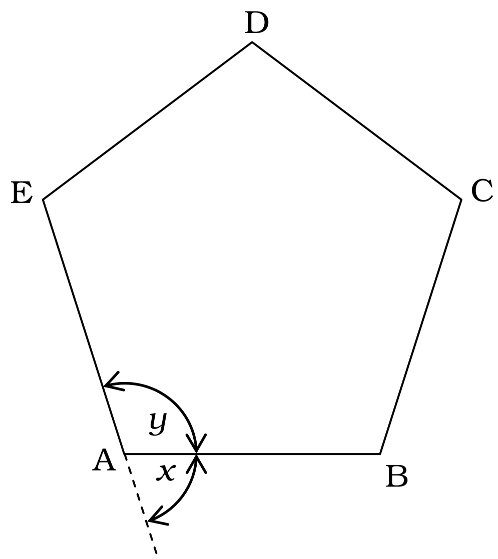


Regular Polygons

A 'regular' polygon has all its sides equal and all its angles equal. A regular polygon with n sides also has n axes of symmetry. We often have a choice of methods when working with regular polygons.

Example 2

Find the size of the interior angles of a regular pentagon.



There are two methods. The first method finds the interior angle, y, directly. Recall that the interior angles of a polygon add up to $(n-2)\times180°$. In the case of a pentagon, n = 5. The interior angles of a pentagon therefore add up to $(5-2)\times180$ = 540°. Now, we are dealing with a regular pentagon which has all its angles equal. The size of each of the five interior angles is therefore $540\div5=108°$.

The second method finds the interior angle, y, indirectly. We start by considering the **exterior** angles of the regular pentagon. The exterior angles of any polygon add up to 360°. The five exterior angles of a regular pentagon are all equal, and must therefore be $360\div5=$ 72°. Now we consider the relationship between an interior angle of any polygon and the corresponding exterior angle. They always add up to 180° because these two angles make a straight line. In the diagram above, $x+y=180$. But x = 72°, so $y=180-72=108°$, as before.

The second method requires two stages compared to the one stage of the first method. However, the theory used by the second method is simpler: there is no need to remember or use the formula $(n-2)\times180$.

The next example requires an approach which is similar to the second method above.

Example 3

An interior angle of a regular polygon is 160°. How many sides does the regular polygon have?

An interior angle and an exterior angle add up to 180°. An exterior angle must therefore be 20°. But the exterior angles of any (convex) polygon add up to 360°. To find the number of exterior angles, perform the division: $360\div20=$ 18. The regular polygon in question must have 18 angles and therefore 18 sides.

A final example introduces another aspect of regular polygons. This Example has several parts, and closely resembles a particular type of examination question. Believe it or not, examiners are sometimes trying to help candidates. In this type of question, the several parts lead up to the final part of the question. The examiners are thus providing a **strategy** that candidates would otherwise have to devise themselves: this can be difficult! However, it is not always easy to see how the earlier parts of a question relate to each other and fit together to achieve the final part of the question.

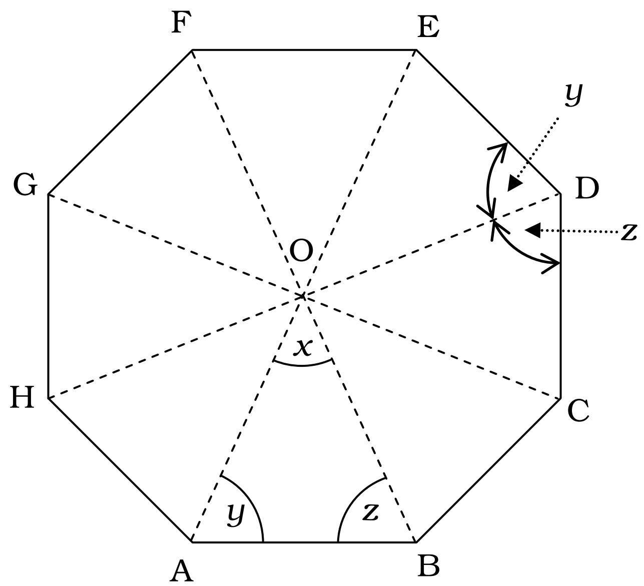
Example 4

ABCDEFGH is a regular octagon.

(a) Find the angle AOB.

(b) Find the angle OAB.

(c) Hence find the size of the interior angle of a regular octagon.



(a) Angle AOB is the angle marked x. ABCDEFGH is a regular octagon, so that x is just one of eight identical angles that fit together round a full circle at O. These eight angles add up to 360°, so each angle must be $360 \div 8 = 45$. So $x = 45°$.

(b) Triangle AOB is isosceles because the octagon is regular. OA must be the same length as OB. The angles opposite, i.e. y and z, must therefore be equal. The angles in triangle OAB must add up to 180°. We know that x is 45°, so the angles y and z must add up to 180 – 45 = 135, and they are equal. $135 \div 2 = 67.5$, so that y and z are both 67.5°.

(c) The interior angle CDE is marked. Notice that this interior angle must be the same as $y + z$, because angle CDO is the same as z, and angle ODE is the same as y. [The octagon is regular, so that the eight isosceles triangles that meet at O, including triangle OAB, triangle OCD and triangle ODE, are all congruent – exactly the same shape and size.]

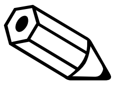
Since y and z are equal, an interior angle of a regular octagon must be $2y$ which is 135°.

Note

1. exterior angle + interior angle = 180°
2. the sum of the exterior angles is 360°.

You can use these properties to calculate the interior angles of a polygon. For example, find the interior angle of a regular hexagon. Since it has six sides and is regular, all the interior angles will be the same. The sum of its six exterior angles is 360°. So each exterior angle is 60°. Therefore each interior angle is 180 – 60 = 120°.

Activity 3 Now tackle these.



1. What is the sum of interior angles of a regular convex polygon with

   (a) 3 sides (b) 4 sides

   (c) 6 sides (d) 8 sides

2. A regular polygon has 20 sides. What is the size of:

   (a) an exterior angle

   (b) an interior angle

3. A regular polygon has exterior angles of 30°. How many sides has it?

4. A regular polygon has interior angles of 135°.

   (a) What is the size of an exterior angle?

   (b) How many sides has it?

5 (a) Find the sum of interior angles of a pentagon.

   (b) Four of the interior angles are 75°, 110°, 124° and 146°. What is the size of the fifth angle?

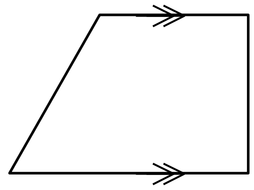
Suggested Answers to Activities

Activity One

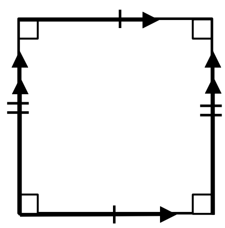
1. $x = 109°$
2. $x = 49°$
3. $x = 76°$, $y = 99°$, $z = 122°$
4. $x = 41°$, $y = 68°$, $z = 83°$
5. $x = 103°$, $y = 73°$
6. $x = 61°$, $y = 119°$, $z = 61°$
7. (a) $x = 63°$, $y = 39°$
   (b) The four angles of the quadrilateral are 73°, 107°, 102° and 78°. These add up to 360°.
8. 4.81 m (3 sf).
9. 53 cm
10. 17 cm

Activity Two

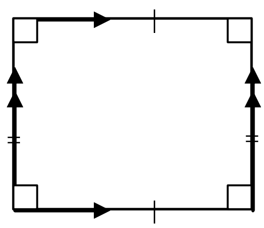
(a) Trapezium



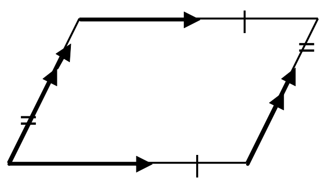
(b) Square



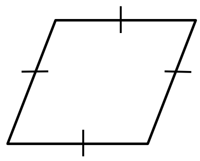
(c) Rectangle



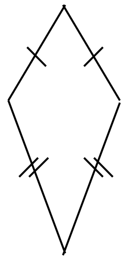
(d) Parallelogram



(e) Rhombus



(f) Kite



Activity Three

1. (a) Size of exterior angles = 360°/3 = 120°
   Size of interior angles = 180° – 120° = 60°
   Sum of interior angles = 3 × 60° = 180°

   (b) 360°
   (c) 720°
   (d) 1080°

2. (a) Exterior angle = 360°/20 = 18°
   (b) Interior angle = 180° – 18° = 162°

3. Number of sides = 360°/exterior angle
   = 360°/30 = 12°

4. Exterior angle = 180° – 135°
   = 45°
   Number of sides = 360°/45° = 8°

5. Exterior angle = 360°/5 = 72°
   Interior angle = 180° – 72° = 108°
   Sum of interior angles = 5 × 108° = 540°
   Size of fifth angle = 540° – (75° + 110° + 124° + 146°)
   = 540° –455° = 85°