Lesson One Numbers

Aims

The aim of this lesson is to enable you to:

- describe and use the number system
- use positive and negative numbers
- work with squares and square roots
- use the sign rule
- master the special language of mathematics

Context

This lesson deals with one of the most fundamental ideas in mathematics – that of number. The lesson is therefore an important introduction to the rest of the module, and indeed the whole course.

You should answer each activity, compare your answers with those at the end of the lesson, and understand any mistakes you may have made, before moving on to the next section of the lesson.

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Introduction

The idea of number is one of the most fundamental in mathematics. Although the basic ideas about number may seem familiar to you, this lesson will introduce you to several new terms and expressions, which you should study carefully and learn thoroughly.

Number Systems

You are familiar with the different types of number that one meets in various situations. These can be classified into various groups, or sets.

The **positive** numbers are those greater than zero. They may be written with a '+' in front, e.g. +5, +1.2, +3/4, or they may be written without a sign, e.g. 3, 199, 5/7.

The **negative** numbers are those less than zero. They are always written with a minus sign, '–', in front, e.g. –4, –0.52, –4/5.

The **natural** numbers, or counting numbers, are: 1, 2, 3, 4, 5...

The **whole** numbers are the natural numbers, together with zero: 0, 1, 2, 3, 4 ...

These are the basic numbers required in counting systems, but as mathematical ideas developed, the need for negative numbers arose, and if we consider the negative as well as positive whole numbers, these are the **integers**: ..., –3, –2, –1, 0, 1, 2, 3, ...

By the introduction of fractions, the number system can be extended, and all numbers that can be expressed in the form $\frac{a}{b}$, where a and b are integers, and b does not equal zero, are called **rational** numbers.

This group will include the integers themselves, since, for example, $5 = \frac{5}{1}$, $-7 = -\frac{7}{1}$, and so the integers can be expressed in the form $\frac{a}{b}$.

Activity 1

Now work through this exercise. The answers are given at the end of the lesson. It is best to write both your rough workings and your answers in the box below, next to the question, wherever you can fit them in. Do *not* send your answers to your tutor.

Use these numbers to answer the questions below:

–1, 5, 1/2, +2, 0, 298, –1/4, –1.25, 1000, +12.

List:

(a) all the natural numbers

(b) all the whole numbers

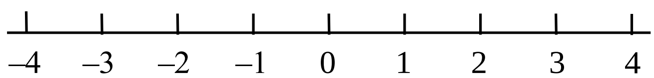
(c) all the positive numbers

(d) all the negative numbers

(e) all the integers

The Number Line

It is often useful to represent numbers diagrammatically, and to do this we use a number line. This is usually drawn horizontally, with the integers marked at equidistant intervals, as indicated:



The convention is always to have the larger numbers to the right, and this is one way of telling which of two numbers is the greater. For instance, –1 is greater than –4, since –1 is to the right of –4 on the number line.

Similarly, –3 is less than 2, since –3 is to the left of 2 on the number line.

Activity 2

1. Draw a number line extending from -10 to +10. Use this number line to identify the greater number in each pair:

   7 or -5
   3 or -6
   -8 or 0
   -4 or -3

2. Write the following list of numbers in order of size, smallest first:

   -4 17 23 -19 -7 11 -9 12 5 -10.

3. Write the following list of numbers in order of size, largest first:

   35 24 -28 -30 -24 0 28 -32.

4. An industrial process can only take place safely when the temperature is greater than -5°C and less than 3°C. On which of the following days can the process take place safely?

Day	Temperature
Monday	-6°C
Tuesday	-1°C
Wednesday	0°C
Thursday	4°C
Friday	-3°C

The number line can also be used to help in adding and subtracting numbers. To add a positive number, move **up** the line (i.e. to the **right**), and to subtract a positive number, move **down** the line (i.e. to the **left**). The process is reversed with negative numbers. So, to add a negative number, move **down** the line (to the **left**), and to subtract a negative number, move **up** the line (to the **right**).

Examples

$-4 + 3 = -1$ (start at –4, move up 3, end at –1)

$2 - 5 = -3$ (start at 2, move down 5, end at –3)
$-1 + (-3) = -4$ (start at –1, move down 3, end at –4)
$-2 - (-5) = 3$ (start at –2, move up 5, end at 3).

Activity 3

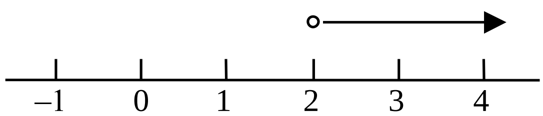
Use the number line to help simplify the following:

(a) $-5 + 8$	(b) $-3 - 6$
(c) $2 + (-4)$	(d) $6 - (-1)$
(e) $-11 + 2$	(f) $4 - 10$
(g) $8 + (-2)$	(h) $-5 - (-2)$

The Number Line: The Numbers in Between

Of course, there are other numbers in between the integers on the number line. These are the fractions and decimals, which are grouped together under the name of **real** numbers.

We can show all the real numbers greater than 2 on the number line like this:

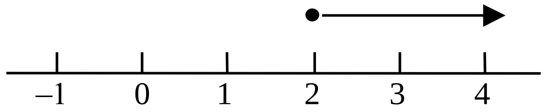


The empty circle over the 2 shows that it is not included. The arrow shows that we can go on forever. The mathematical way of writing

any number greater than 2

is $x > 2$

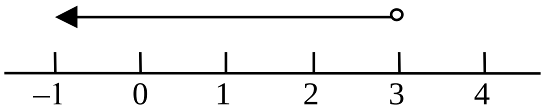
To show that 2 *is* included, we put a filled-in circle over the 2:



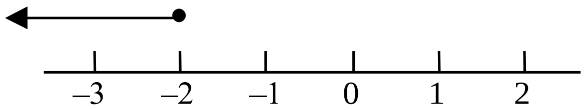
Similarly, the mathematical expression '$x \geq 2$' means 'x is greater than or equal to 2'.

To show numbers *less than* a certain amount, the signs are reversed. For example,

$$x < 3$$

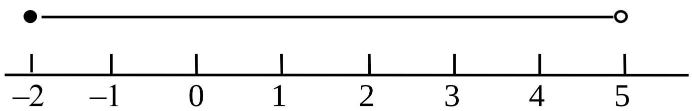


$$x \leq -2$$



To give an upper and lower limit of the numbers, both signs are used together with 'x' in the middle. For example,

$$-2 \leq x < 5$$



Practise reading this expression aloud: 'x is greater than or equal to minus two and less than five'.

However, pairs of inequalities cannot always be combined. In the case just considered, the two inequalities are defining a **single** region of the number line.

Consider the inequalities $x \leq -2$ and $x > 5$. If we put these together, with x in the middle, then the result is $5 < x \leq -2$. (Note that the second inequality has first been rewritten as 5 < x). Now this does not make sense! 5 is not less than -2, so something has gone wrong. In this case, the two inequalities describe two **separate** regions of the number line, and therefore cannot be 'combined'.

Activity 4

1. Write these expressions using mathematical symbols. Be very careful to get the 'greater than' and 'less than' signs the right way round.

   (a) x is less than minus one

   (b) y is greater than or equal to seventeen

   (c) x is less than or equal to ten and greater than five

   (d) $3x$ is greater than one and less than one hundred

   (e) $x + y$ is less than five and greater than or equal to minus seven.

2. Illustrate the following on separate number lines:

   (a) $-4 \leq x < 2$

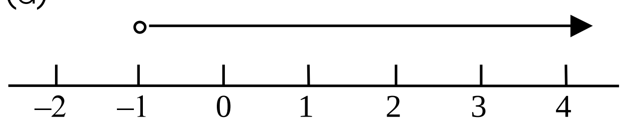
   (b) $x \leq -1$

   (c) $3 < x < 5$

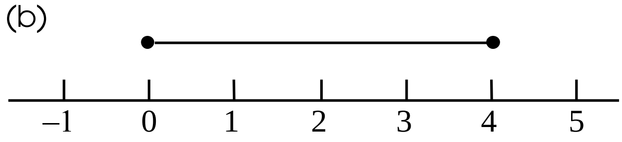
   (d) $x > 2$

3. Write symbolically (i.e. as the questions in (ii) are written) the following diagrammatic illustrations:

   (a)



   (b)



Positive and Negative Numbers

We are all familiar with using positive numbers in everyday life, for measuring, counting, shopping and the like. Now we are going to explore further how to use negative numbers.

Log on to Twig and look at the film titled: **India and Negative Numbers**

www.ool.co.uk/1742ew

Discover when and why negative numbers were developed, and the simple rules that govern their use.

Using the number line, we have seen how to add and subtract using positive and negative numbers. Did you notice that when you subtracted a negative number, you moved up the number line, in the same way you would move for an addition? In other words, two negatives made a positive. We could write it like this:

$$\begin{aligned} 3-(-2) &= 3+2 \\ &= 5 \end{aligned}$$

A useful way to understand this is to imagine that four bills, each for £50, were delivered to your house. Then the postman came and told you that it was a mistake, so you were £200 better off than you thought you were! We could say each bill was –£50, since it is money you have to give away (alas). So we could write it like this:

$$\begin{aligned} -(-£50)-(-£50)-(-£50)-(-£50) &= £50+£50+£50+£50 \\ &= £200 \end{aligned}$$

When you multiply two negative numbers, the result is much the same. (After all, multiplying is just a quick way of adding, e.g. $4 \times 50 = 50 + 50 + 50 + 50$). So, going back to those wrongly delivered bills again, we can say $(-4) \times (-£50) = +£200$. The same is true for dividing.

This is the first pattern you need to remember:

$$\begin{aligned} \text{positive} \times \text{positive} &= \text{positive} \\ \text{negative} \times \text{negative} &= \text{positive} \\ \text{positive} \div \text{positive} &= \text{positive} \\ \text{negative} \div \text{negative} &= \text{positive} \end{aligned}$$

'A positive and a positive or a negative and a negative make a positive.'

Be careful. When only one of the terms is negative, then the answer stays negative. So if you get a bill for £25 on Monday, two bills for £25 on Tuesday and another bill for £25 on Wednesday, your financial situation is $4 \times (-£25) = -£100$. If you decided to pay half your bills this week, you would still owe half the money, so the answer is still negative:

$$(-£100) \div 2 = -£50 .$$

So this is the pattern to remember:

$$\begin{aligned} \text{positive} \div \text{negative} &= \text{negative} \\ \text{negative} \div \text{positive} &= \text{negative} \\ \text{positive} \times \text{negative} &= \text{negative} \\ \text{negative} \times \text{positive} &= \text{negative} \end{aligned}$$

'A positive and a negative make a negative.'

These are fundamental principles of number work and you need to know these patterns very well.

$$\begin{aligned} 3 - (-2) &= 3 + 2 \\ &= 5 \end{aligned}$$

It is also possible to perform calculations with negative numbers on a scientific calculator. It is important to note that there are two different types of minus sign is use:

- To indicate the operation of subtraction
- To indicate that a number is negative

The **-** button performs subtraction. Modern Casio calculators have a **(-)** button to make a number negative, while older scientific calculators have a **+/-** button. The difference is not just in appearance, however. The **(-)** button comes **before** the number it applies to, exactly as the sum is written. The +/- button, however, comes **after** the number it applies to.

So the above calculation on a modern Casio calculator would be performed:

3 – (-) 2 =

The same calculation on an older scientific calculator would be:

3 – 2 +/- =

Log on to Twig and look at the film titled: **The Discovery of Zero**

www.ool.co.uk/1671dj

For centuries, only the numbers 1-9 existed. Who discovered zero? When? And why is it important?

Important Facts

The following important facts are often overlooked. They are useful not just for arithmetic but also in the algebra which occurs later in the course.

1. Adding zero has no effect:

$9714 + 0 = 9714$
$0 + 682 = 682$
$(-15) + 0 = (-15)$

2. Multiplying by zero always gives an answer of zero:

$5703 \times 0 = 0$
$0 \times 146 = 0$
$0 \times (-28) = 0$

3. Zero divided by any number (other than zero) is still zero:

$0 \div 30951 = 0$
$0 \div (-94) = 0$

4. Dividing by zero is not possible:

$1 \div 0$ has no meaning. Try it on a calculator!

5. Multiplying a number by one has no effect:

$$748321 \times 1 = 748321$$
$$1 \times 2501 = 2501$$
$$(-550) \times 1 = (-550)$$

Summary of Sign Rules

The same rule applies to both multiplication and division of two numbers:

The two input numbers	**Output**
Signs the same: ++ or --	+
Signs different: + - or -+	-

Example

Find (a) 6×-7 (b) $7 \times (-6)$ (c) $(-6) \times (-7)$

First, consider the simpler multiplication: $6 \times 7 = 42$. All three answers to this example will therefore be either 42 or -42.
Next consider the signs. For (a) and (b) the signs are different (+- and -+ respectively). So the answers to (a) and (b) are both -42.

For (c), the signs are the same (--), so the result is positive: 42.

Activity 5

1 Do the following multiplications mentally:

(a) $(-2) \times 8$ (b) $(-6) \times (-4)$
(c) $5 \times (-2)$ (d) $(-8) \times (-8)$

2 Do the following divisions mentally:

(a) $6 \div (-3)$ (b) $(-15) \div 5$
(c) $(-4) \div 4$

3 Write down the answers to the following:

(a) $1 \times (-53)$ (b) 6580×1
(c) $0 + 624$ (d) $(-76) + 0$
(e) 935×0 (f) $0 \times (-47)$
(g) $0 \div (-8203)$ (h) 1×0

4. Do the following multiplications mentally:
   (a) $6 \times (-8)$ (b) $(-6) \times (-8)$
   (c) $(-6) \times 8$ (d) $(-7) \times 9$
   (e) $7 \times (-9)$ (f) $(-7) \times (-9)$

5. Do the following divisions mentally:
   (a) $63 \div (-7)$ (b) $(-63) \div 7$
   (c) $(-63) \div (-7)$

6. (a) The temperature is -8°C. How many degrees must it rise to become +7°C?

   (b) The temperature is 12°C and it falls to -3°C overnight. Through how many degrees has it fallen?

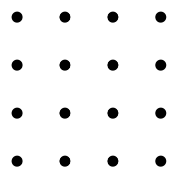
   (c) After rising 13°C, the temperature is +1°C. What was the temperature originally?

Further Types of Number

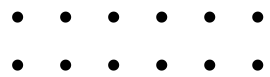
Often the whole numbers, or integers, are subdivided into **groups** satisfying certain conditions. For instance, the whole numbers can be separated into the even numbers, 0, 2, 4, 6, 8, ... and the odd numbers, 1, 3, 5, 7, 9, ...

Another type of number is a **square** number, i.e. a number that is formed by 'squaring' another number, or multiplying that other number by itself. E.g. 25 is a square number, since $25 = 5 \times 5$ or 5^2, where 5^2 is read as '5 squared'. If we restrict ourselves to the natural numbers, those which are square are 1, 4, 9, 16, 25, ... Note that no negative number can be a square number, since, whether a number starts as positive or negative, when multiplied by itself, it is always positive, e.g. $(-3) \times (-3) = +9$.

Square numbers can be represented by an array of dots that themselves form a square, e.g. 16 can be represented thus:

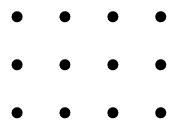


Using this idea, we can also have **rectangular** numbers, where the numbers are represented by a rectangular array of dots, where there must be at least two in both length and the breadth. For instance, 12 is rectangular since it can be represented either by



(6×2)

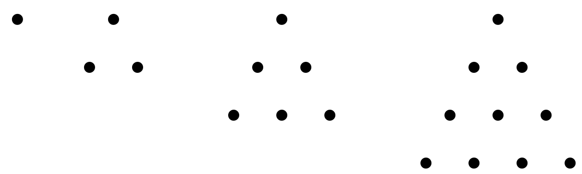
or



(4×3)

but 5 is not rectangular, since the only way to represent 5 would be

Triangular numbers are those which form a triangular array of dots, with the same number of dots along each side, thus:



etc., and, counting the dots, we can see that these will be the numbers 1, 3, 6, 10.

Similarly, we could have the group of cubes, or numbers multiplied by themselves three times. Hence

$$\begin{aligned} 2^3 &= 2 \times 2 \times 2 \\ &= 8 \end{aligned}$$

You can say either 'two cubed' or 'two to the power of three'.

It is very useful to recognise some cubes – you could learn these off by heart:

$$2^3 = 8$$

$$3^3 = 27$$

$$4^3 = 64$$

$$5^3 = 125$$

Don't forget that we can also reverse the process and talk about the **square root** of a number. If $5^2 = 25$, the square root of 25 is 5, and we write this using the square root sign: $\sqrt{\ }$, e.g. $\sqrt{25} = 5$ ('the square root of 25 equals 5').

The cube root uses the same sign with a little 3 above it: e.g. $\sqrt[3]{27} = 3$ ('the cube root of 27 equals 3').

Log on to Twig and look at the film titled: **Number Theory**

www.ool.co.uk/1755bv

How Gauss found the sum of all the numbers between 1 and 100, in a matter of seconds, aged just seven! He had developed a formula for triangular numbers, and went on to be hailed as the Prince of Mathematics as he moved on to more complex problems in later life.

Squares and Square Roots

Consider the operation of 'squaring':

$1 \rightarrow 1$
$2 \rightarrow 4$
$3 \rightarrow 9$
$4 \rightarrow 16$
$5 \rightarrow 25$
etc.

The reverse operation is called 'square rooting', and simply reverses the direction of the arrows above:

$\pm 1 \leftarrow 1$
$\pm 2 \leftarrow 4$
$\pm 3 \leftarrow 9$
$\pm 4 \leftarrow 16$
$\pm 5 \leftarrow 25$
etc.

Note that the two statements:

$9^2 = 81$
$\sqrt{81} = \pm 9$

are equivalent: they are two different ways of saying the same thing.

Scientific calculators have both 'square' $\boxed{x^2}$ and 'square root' buttons: $\boxed{\sqrt{\ }}$.Without a calculator, squaring is straightforward: multiply the number by itself. So, to find 7^2, simply do $7 \times 7 =$ 49. Finding a square root is less straightforward without a calculator. To find, for example, the square root of 81, it is necessary to find a number which, when multiplied by itself, gives 81. If the answer does not spring immediately to mind, then trial and error will be required. Make an initial guess of 5: 5^2 is 25: not large enough. Similarly for $6^2 = 6 \times 6 = 36$, $7^2 = 7 \times 7 = 49$ and $8^2 = 8 \times 8 = 64$. However, $9^2 = 9 \times 9 = 81$, so that the square root of 81 is ± 9.

It is only reasonable to ask about square roots of numbers in between, such as $\sqrt{80}$, for example. The answer is not a whole number: it is a decimal that continues for ever and ever without any pattern. If we did need to know an approximate decimal value of this square root, then a calculator would normally be used. There is a complicated written method that used to be taught in schools, but you will be reassured to hear that there is little chance of its revival.

If you need to find a square root on a calculator, make sure you know how your calculator works. On older scientific calculators, the square root button is pressed **after** the number: for example:

81 $\boxed{\sqrt{\ }}$

On modern calculators, the square root button is pressed **before** the number, exactly as written. Also, note that the [=] button is required (if no other numbers are involved in the calculation):

[√] 81 [=]

Activity 6

1. From the set of numbers

   8, 17, 25, 35, 45, 64, 81

   write down

   (a) all the square numbers

   (b) all the cube numbers

   (c) all the odd numbers

   (d) all the even numbers

2. Work out the following:

   (a) 3^2

   (b) Seven squared

   (c) 6^3

   (d) $\sqrt{100}$

Square Roots and Negative Numbers

We have seen already that $(-3) \times (-3) = +9$. We must also remember that this works the other way as well. The square root of +9 could be +3, but it could also be -3. Thus every positive number has *two* square roots (the same number but with the sign reversed) and, when in doubt, you should always put down *both* of these possibilities!

Squares, Cubes and Mental Arithmetic

A calculator may not always be available and you should memorise squares up to 15^2 and cubes up to 5^3. These are:

$1^2 = 1$	$2^2 = 4$	$3^2 = 9$	$4^2 = 16$
$5^2 = 25$	$6^2 = 36$	$7^2 = 49$	$8^2 = 64$
$9^2 = 81$	$10^2 = 100$	$11^2 = 121$	$12^2 = 144$
$13^2 = 169$	$14^2 = 196$	$15^2 = 225$	
$1^3 = 1$	$2^3 = 8$	$3^3 =27$	$4^3 = 64$
$5^3 = 125$	$10^3 = 1000$		

You should also be able to work backwards comfortably and recall that 81's square root is 9 (or -9) and 64 has a square root of 8 (or -8) and a cube root of 4.

The Language of Maths

There are many words which are used in special ways by mathematicians. We will try to define the important ones as we go along. Here are four words you will see often:

Word	*Definition*	*examples*
Equation	A statement that two mathematical (or logical) expressions are equal to one another. An algebraic equation is one in which unknown quantities are normally represented by letters, e.g. x and y.	$2 + 3 = 5$ beauty is truth $x^2 - 5x + 4 = 0$ $x + y = 5$
Formula	An equation which describes the relationship between two or more quantities.	$e = mc^2$ $I = PRT$
Identity	An algebraic equation that remains true whatever values are substituted for the symbols.	$(x^A)^2 = x^{2A}$ $(x – y)^2 = x^2 - 2xy + y^2$

Expression	A symbol or combination of symbols which serve to describe a quantity.	$5x^2$ $x + 2y$ $(x - y)^2$ $a + 2b - 3c + 4d - 5e$

Be careful to use these terms correctly (although, as you can see, 'equation' and 'formula' are often very similar in meaning).

Suggested Answers to Activities

Activity One

(a) 5, +2, 298, 1000, +12

(b) 0, 5, +2, 298, 1000, +12

(c) 5, 1/2, +2, 298, 1000, +12

(d) –1, –1/4, –1.25

(e) –1, 5, +2, 0, 298, 1000, +12

Activity Two

1. (a) 7 (b) 3 (c) 0 (d) -3

2. -19, -10, -9, -7, -4, 5, 11, 12, 17, 23.

3. 35, 28, 24, 0, -24, -28, -30, -32.

4. Tuesday, Wednesday, Friday

Activity Three

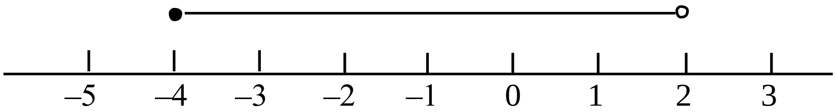
(a) 3
(b) –9
(c) –2
(d) 7
(e) –9
(f) –6
(g) 6
(h) –3

Activity Four

1. (a) $x < -1$
   (b) $y \geq 17$

(c) $5 < x \leq 10$
(d) $1 < 3x < 100$
(e) $-7 \leq x + y < 5$

2. (a)



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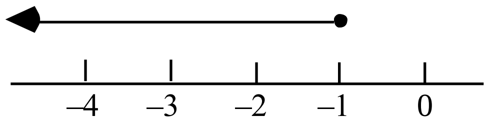

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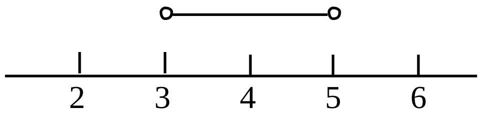

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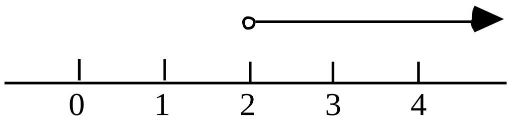

3. (a) $x > -1$ (b) $0 \leq x \leq 4$.

Activity Five

1 (a) –16 (b) 24 (c)–10 (d) 64

2 (a) –2 (b) –3 (c) –1

3 (a) –53 (b) 6580 (c) 624 (d) –76
(e) 0 (f) 0 (g) 0 (h) 0

4 (a) –48 (b) 48 (c) –48 (d) –63
(e) –63 (f) 63

5 (a) –9 (b) –9 (c) 9

6 (a) 15°C (b) 15°C (c) –12°C

Activity Six

1. (a) 25, 64, 81 (b) 8, 64
(c) 17, 25, 35, 45, 81 (d) 8, 64

2. (a) 9 (b) 49 (c) 216 (d) ±10