Lesson Four

Straight Lines

Aims

The aims of this lesson are to enable you to

- find the distance between two points
- find mid–points of two points
- find the equation of a straight line
- find the gradient of a line perpendicular to a given line
- find the intersection of two lines

Context

The techniques covered in this lesson are the basic tools needed for dealing with many geometrical problems.

References are made in this lesson to Bowles, Ch. 1.6.

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The Distance between Two Points

The distance between two points can be found by applying Pythagoras' Theorem.

Example 1

Find the distance between the points (1,2) and (5,5).

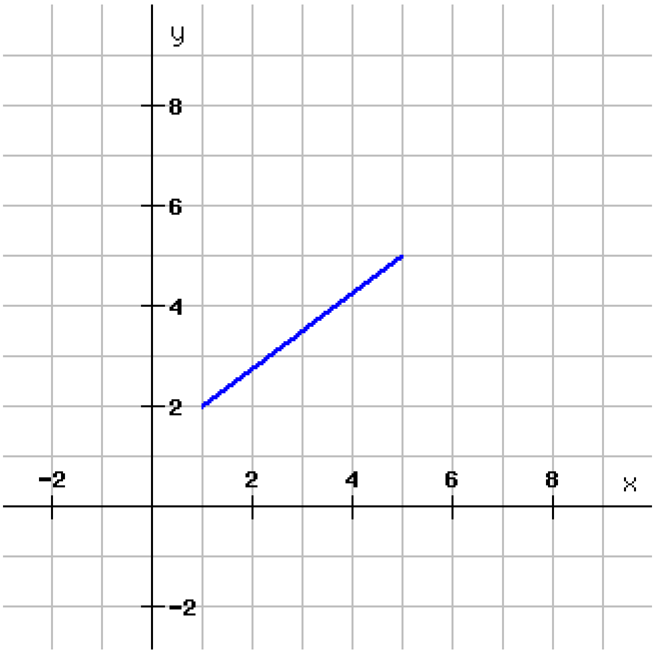

Figure 1

In going from (1,2) to (5,5), the x-coordinate increases by 4 and the y-coordinate increases by 3.

The distance between the two points is therefore $\sqrt{4^2 + 3^2} = 5$.

Example 2

Find the distance between the points (4,–2) and (–8,–7)

Because we are only interested in the square of the difference between the x-coordinates (and similarly for the y-coordinates), it doesn't matter whether we calculate the distance as

$$\sqrt{(4 - [-8])^2 + (-2 - [-7])^2}$$

or $\sqrt{(-8 - 4)^2 + (-7 - [-2])^2}$

or even $\sqrt{(-8-4)^2 + (-2-[-7])^2}$

In each case the answer is $\sqrt{12^2 + 5^2} = 13$

Mid–points

Considering the points (1,2) and (5,5) again, suppose that we are now interested in the point halfway between them. Technically this is described as 'the mid–point of the line segment joining the points (1,2) and (5,5)'. (The 'line segment' starts at (1,2) and ends at (5,5).)

In the diagram below, B' is the required mid–point, so that AB' is half of AB.

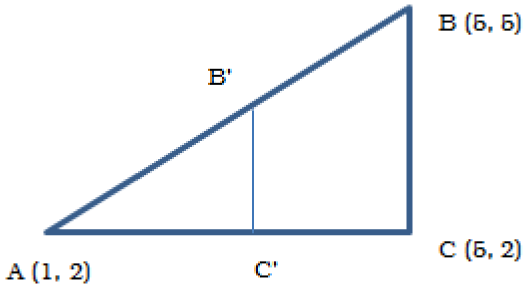

Figure 2

Now consider the triangles ABC and A$B'C'$. These both have the same angles, and they are referred to as 'similar triangles'. This means that their sides are in the same proportion. So, if AB' is half of AB, it follows that AC' is half of AC and that $B'C'$ is half of BC.

This makes it possible to work out the coordinates of B':

As C' is halfway between A and C, the x coordinate of B' is the average of 1 and 5 (the x-coordinates of A and C); i.e. ½ (1 + 5) = 3

And, in the same way, as the y-coordinate of B' is halfway between the y-coordinates of C and B, the y-coordinate of B' is the average of 2 and 5; i.e. ½ (2 + 5) = 3.5

Example 3

Find the mid–point of the line segment joining the points (–5,4) and (–2,–4).

This can be found without even considering the relative positions of the two points, since all we need to do is find the average of the two x-coordinates and the average of the two y-coordinates.

The mid–point is thus ($\frac{1}{2}$ (–5 + [–2]), $\frac{1}{2}$ (4 + [–4]) = (–3.5, 0)

Example 4

If M is the mid–point of the line segment PQ, where P is the point (2,3) and M is the point (4,8), find the coordinates of Q.

Let the coordinates of Q be (a,b).

Then $\frac{1}{2}(2 + a) = 4$ and $\frac{1}{2}(3 + b) = 8$.

This gives $a = 6$ and $b = 13$, so that the coordinates of Q are (6,13).

Activity 1

Find the distances between the following pairs of points, and also the mid–points of the line segments joining them.

(i) (1,3) and (7,11)

(ii) (2,5) and (5,10)

(iii) (3,–1) and (–2,4)

(iv) (–4,–6) and (7,–12)

The Gradient of a Straight Line

The gradient of a line refers to its slope, which can be defined as

$$\frac{\text{distance up}}{\text{distance along}}$$

where it is assumed that we are moving from left to right (so that x is increasing).

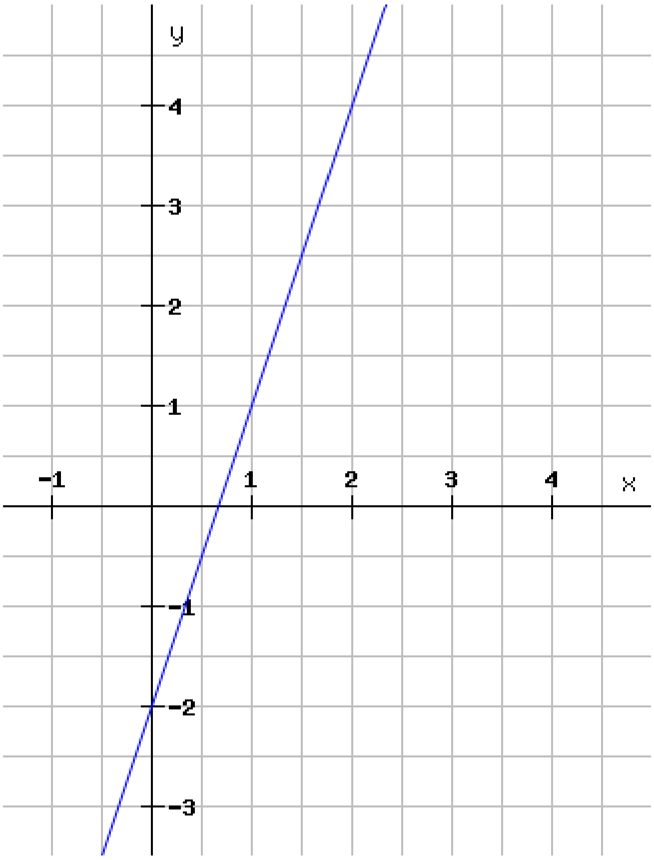

Figure 3

To work out the gradient of the above line, we can start at any point on the line – for example, (1,1). We can then move along by any convenient amount in the x direction. In this example, it is simplest to go along just 1, giving:

$$\frac{\text{distance up}}{\text{distance along}} = \frac{3}{1} = 3$$

Had we been so inclined we could have started at (0,–2) and gone along 2, and obtained

$$\frac{\text{distance up}}{\text{distance along}} = \frac{6}{2} = 3$$

Example 5

Find the gradient of the line shown below:

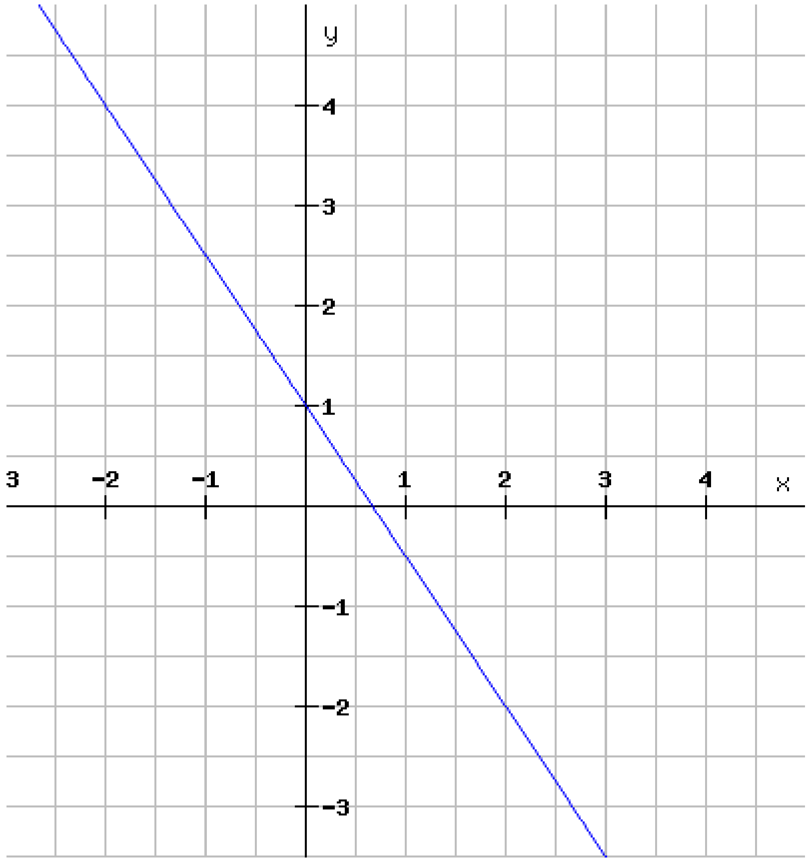

Figure 4

We could start at (0,1), for example, and go along 2, but as the line slopes downwards the 'distance up' is now negative.

So the gradient = $\frac{-3}{2}$

The Equation of a Straight Line – Part 1

Unless a straight line is vertical, it is the graph of a function of x (see Lesson One): for every value of x there will be a single corresponding value of y.

By the expression 'equation of a straight line' we mean this function, an example of which would be $y = 2x + 1$, which is shown below, in Figure 5.

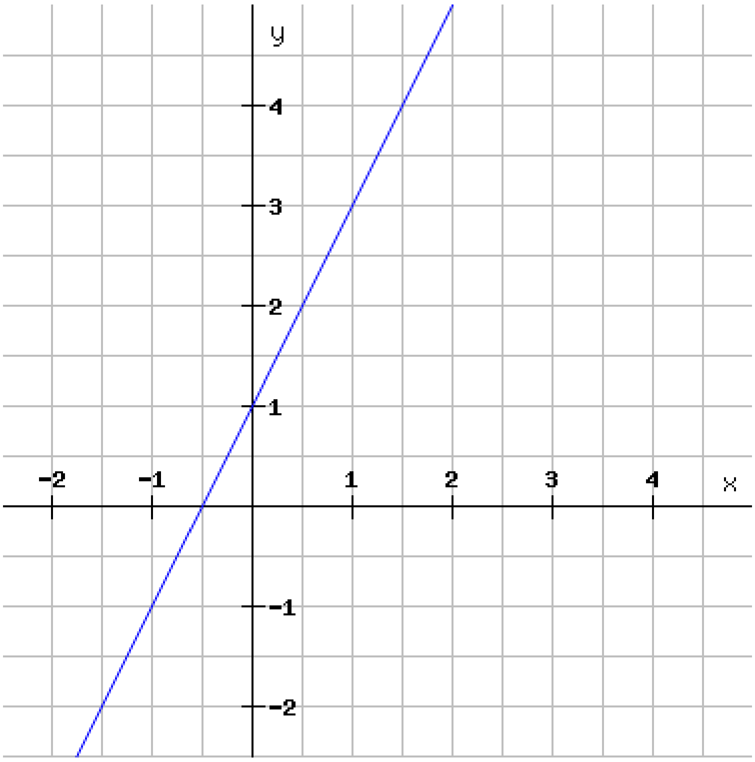

Figure 5: $y = 2x + 1$

Note 1: Strictly speaking, we should talk about a 'straight line function', rather than 'equation'. However the terminology is too well established for it to be changed.

Note 2: A 'straight line' is often just referred to as a 'line'. If the line wasn't straight it would be referred to as a 'curve'.

We can make a couple of observations about the line $y = 2x + 1$. First of all, if $x = 0$ then $y = 1$, and the line crosses the y–axis when $y = 1$.

Also, as we increase x by 1, y increases by 2. In other words, the gradient of the line is 2.

In general, a line of the form $y = mx + c$ can be described as having a 'y–intercept' of c and a gradient of m. (This may be familiar to you from GCSE.)

Equations of lines need not be given in the form $y = mx + c$:

Example 6

Find the gradient and y–intercept of the line $2x + 3y + 4 = 0$.

Rearranging, we have $3y = -2x - 4$ and $y = -\frac{2}{3}x - \frac{4}{3}$

Whilst the form $2x + 3y + 4 = 0$ is neater, the $y = mx + c$ form tells us that the gradient is $-\frac{2}{3}$ and that the y–intercept is $-\frac{4}{3}$. This is confirmed by the graph of the line (shown below in figure 6), which slopes downwards. Note that a gradient of -1 would mean a downwards slope of 45^0, and our line has a downwards slope of less than 45^0.

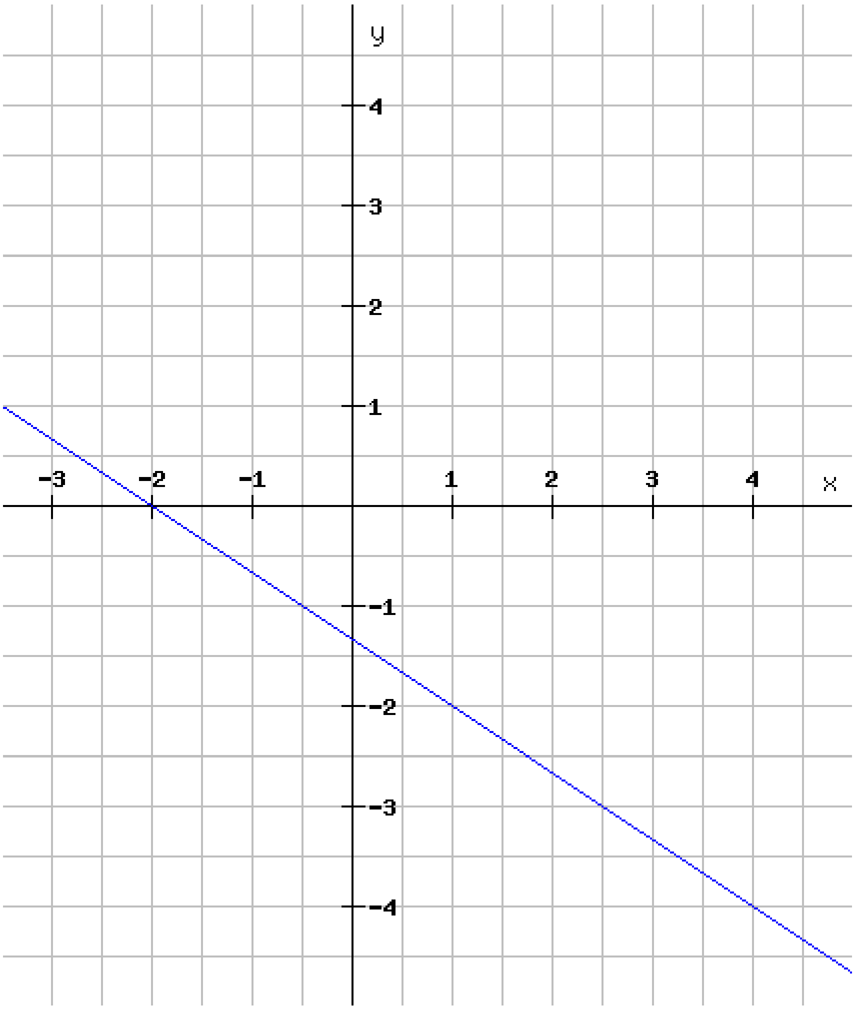

Figure 6: $2x + 3y + 4 = 0$

At A-level you are encouraged to sketch graphs based on their important features, such as gradients and where the graph crosses the x and y–axes.

In the above example, an alternative to rearranging the equation in order to find its gradient is to note the following:

When $x = 0$, $2x + 3y + 4 = 0$ becomes $3y + 4 = 0$, which gives $y = -\frac{4}{3}$. This is the y–intercept.

Then, when $y = 0$, $2x + 3y + 4 = 0$ becomes $2x + 4 = 0$, which gives $x = -2$. This can be termed the x–intercept.

Plotting the two points $(0,-\frac{4}{3})$ and $(-2,0)$ enables the graph to be drawn.

Activity 2	Draw the graphs of the following lines by finding the x and y–intercepts. Then rearrange the equations into the form $y = mx + c$ and check that the gradient (m) agrees with the graph.
	(i) $4x + 2y - 1 = 0$ (ii) $5 = 3x + 4y$ (iii) $x = 3 - 2y$

The Gradients of Parallel and Perpendicular Lines

By definition, lines that are parallel have the same slope and therefore have the same gradient.

Horizontal and vertical lines need to be considered separately:

For example, the line $y = 2$ could be written as $y = 0x + 2$. In this form, it is clear that it has a gradient of 0 (the line goes up 0, for every 1 across) and is therefore horizontal.

However, the line $x = 2$ is different, and cannot be written in the form $y = mx + c$.

By comparing it with the line $y = 2$, however, we can see that $x = 2$ is a vertical line, with infinite gradient.

A useful result can be found connecting the gradients of perpendicular lines (i.e. lines that are at right angles to one another).

Consider for example the line $y = 2x$. This has a y–intercept of 0, and therefore passes through the point (0,0). It is fairly easy to draw in a line perpendicular to it, as shown below in figure 7.

Whereas the line $y = 2x$ goes up 2 for every 1 across, the perpendicular line goes down 1 for every 2 across. This can be expressed as going up –1 for every 2 across, or up $-\frac{1}{2}$ for every 1 across. In other words, its gradient is $-\frac{1}{2}$.

This reasoning also applies to the line $y = \mathrm{m}x$, and indicates that, in general, a perpendicular line has a gradient of $-\frac{1}{m}$. (The same argument applies to a general line $y = \mathrm{m}x + \mathrm{c}$.)

So, if two lines with gradients m_1 and m_2 are perpendicular, then

$$m_2 = -\frac{1}{m_1} \quad \text{or} \quad m_1 = -\frac{1}{m_2}$$

or alternatively $m_1 m_2 = -1$

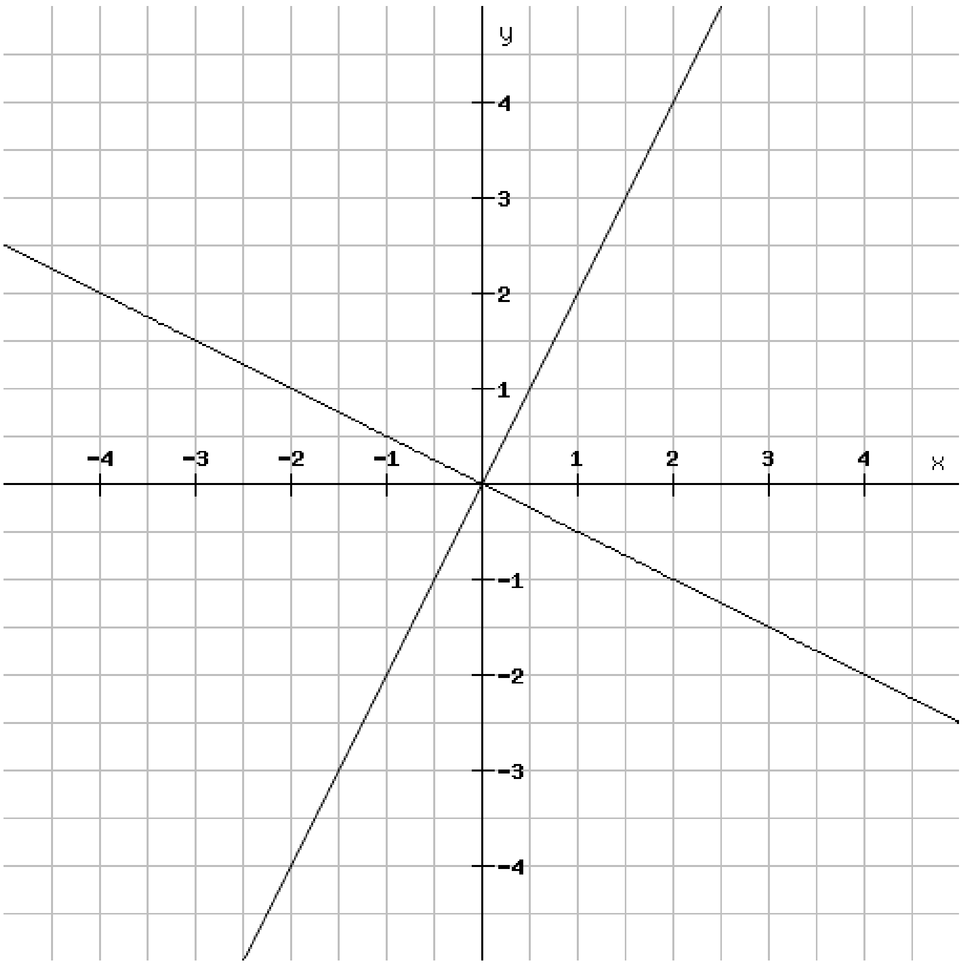

Figure 7: $y = 2x$ and $y = -\frac{1}{2}x$

Activity 3	Which of the following lines are parallel to $x + 2y = 1$, and which are perpendicular to it?
	(a) $y = 2x + 1$ (b) $x + 2y = 3$ (c) $2x + 4y = 5$ (d) $2x + y = 1$ (e) $3 + 4x - 2y = 0$

The Equation of a Straight Line – Part 2

So far we have been able to construct the equation of a line from its gradient and the y-intercept, and vice-versa.

We have also used the equation of a line to find the points where it crosses the x and y-axes.

Another possibility is that we may be given two points on the line (which may or may not be the points where it crosses the x and y-axes). We may then be interested in finding the gradient and the equation of the line.

Consider the example in figure 8. The line is known to pass through the points (2,3) and (4,9). The point (x,y) could be anywhere on the line (it needn't be between the two given points).

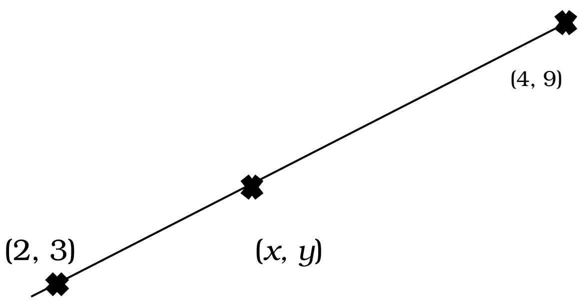

Figure 8

Using the definition of the gradient as $\frac{\text{distance up}}{\text{distance along}}$, we can obtain two expressions for the gradient, m:

$m = \frac{9-3}{4-2}$ and $m = \frac{y-3}{x-2}$ (alternatively, we could have used $m = \frac{y-9}{x-4}$)

So, if all we need to find is the gradient, then we have $m = \frac{6}{2} = 3$.

If we had been told that the gradient was 3 and just given the point (2,3), then we could write

$3 = \frac{y-3}{x-2}$ so that $y - 3 = 3(x - 2)$ and $y = 3x - 3$

Finally, if we are just given the two points (2,3) and (4,9), we have:

$\frac{9-3}{4-2} = \frac{y-3}{x-2}$, which gives $3 = \frac{y-3}{x-2}$ and $y = 3x - 3$, as before.

Example 7

Find the equation of the line passing through the points (2,–1) and (–3,4)

First of all, the gradient of the line is $\frac{-1-4}{2-(-3)} = \frac{-5}{5} = -1$

Note: It doesn't matter which way round the points are. We could also have written

$\frac{4-(-1)}{-3-2} = \frac{5}{-5} = -1$ (by changing the order of the points, we are just multiplying the top and bottom by –1).

We then introduce x and y into the equation by finding another expression for the gradient.

We could, for example, use the point (2,–1) to give $\frac{y-(-1)}{x-2}$ as the gradient.

So $-1 = \frac{y-(-1)}{x-2}$ and hence $-x + 2 = y + 1$, so that $y = 1 - x$ (this is a slightly more elegant form than $y = -x + 1$).

Alternatively, we could have used the point (–3,4), to give $-1 = \frac{y-4}{x-(-3)}$

and $-x - 3 = y - 4$ and $y = 1 - x$ again.

Example 8

Find the equation of the line passing through the point (1,4) with gradient 2.

$\frac{y-4}{x-1} = 2$, so that $y - 4 = 2x - 2$ and $y = 2x + 2$

An alternative method is as follows:

Let the line have equation $y = 2x + c$

Then, as the point (1,4) has to satisfy the equation, we have $4 = (2 \times 1) + c$, which gives $c = 2$.

Example 9

Find the equation of the line passing through the point (0,2) with gradient 3.

Note that this can be written down straightaway as $y = 3x + 2$, since (0,2) is just the y–intercept.

However, the method of the previous examples is also valid, and we could write $3 = \frac{y-2}{x-0}$, which gives $y = 3x + 2$ as well.

Example 10

Find the equation of the line passing through the points (0,1) and (4,0)

Equating two expressions for the gradient gives

$\frac{y-1}{x-0} = \frac{1-0}{0-4} = \frac{1}{-4}$, so that $y = 1 - \frac{x}{4}$

It is a good idea to check that both the points (0,1) and (4,0) satisfy this equation:

$1 = 1 - \frac{0}{4}$ and $0 = 1 - \frac{4}{4}$

A general formula can be derived as follows. If the line passes through the points (x_1,y_1) and (x_2,y_2), then equating the two expressions for the gradients gives

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$ (A)

If we are told that the gradient is m, and that the point (x_1,y_1) lies on the line, then (A) becomes:

$\frac{y-y_1}{x-x_1} = m$, which can also be written as $y - y_1 = m(x - x_1)$.

Activity 4

Find the equations of the following lines:

(i) The line passing through the points (3,1) and (7,3)

(ii) The line passing through the points (–4,2) and (1,–1)

(iii) The line with gradient 2 that passes through the point (4,–2)

(iv) The line with gradient –1 that passes through the point (0,1)

(v) The line with gradient 4 that passes through the point (1,0)

The Intersection of Two Lines

To find the coordinates of the point where two lines meet, we need to solve two simultaneous equations. This topic was covered in Lesson Two.

Example 11

Find the coordinates of the point of intersection of the lines $x + 2y = 3$ (A) and $3x - y = 2$ (B).

The equations of the two lines have been given in the standard form for simultaneous equations. In this case we could, for example, multiply equation (B) by 2 and add it to (A), to give:

$7x = 7$, so that $x = 1$, and then either (A) or (B) gives $y = 1$.

As the question asks for the coordinates of the point of intersection, the answer is (1,1).

Example 12

Find the coordinates of the point of intersection of the lines $y = 2x + 1$ and $y = 3x - 1$

Here the equations are in a more convenient form. Equating the two expressions for y, we have:

$2x + 1 = 3x - 1$, so that $x = 2$, and then $y = 5$. The coordinates of the point of intersection are therefore (2,5).

Example 13

Find the coordinates of the point of intersection of the lines $2x - y - 8 = 0$ and $x - 2y - 7 = 0$

Method 1 We could rearrange the equations into the standard form for simultaneous equations, to give:

$2x - y = 8$ (A) and $x - 2y = 7$ (B)

Then, for example, (A) – 2(B) gives:

$-y - 2(-2y) = 8 - 2(7)$, so that $3y = -6$ and $y = -2$, and (B) gives $x = 3$.

Therefore the coordinates of the point of intersection are $(3,-2)$.

Method 2 The second equation can be rearranged to give $x = 2y + 7$.

Substitution into the first equation then gives $2(2y + 7) - y - 8 = 0$ so that $3y + 6 = 0$ and $y = -2$ again etc.

Note: Having found the point of intersection of two lines, it is a good idea to substitute the coordinates into the equations of both lines, to ensure that they fit.

Activity 5	In each case, find the coordinates of the point of intersection of the given lines:
	(i) $y = 2x + 5$ and $y = 3x + 5$ (ii) $y = 6 - 2x$ and $y = 3x + 16$ (iii) $x + y = 1$ and $3x - 2y = 8$ (iv) $4x + 7y - 2 = 0$ and $x - y + 5 = 0$

Applications to Geometry

Given the coordinates of two points, the following can be determined:

- the distance between the points
- their mid–point
- the gradient of the line passing through the two points
- the equation of the line passing through the two points

When faced with a geometrical problem involving points or lines – in particular, parallel or perpendicular lines, consider which of the above methods are relevant.

Example 14

Show that the points (0,2), (3,4), (5,1) and (2,–1) are the four corners of a square.

In this case we can find the lengths of each of the sides and show that they are equal. Then, to prove that the shape is a square (rather than a rhombus), we could find the gradients of two adjacent sides and demonstrate that the sides were perpendicular.

Example 15

Find the equation of the perpendicular bisector of the points (1,4) and (5,14).

First of all we find the mid–point of (1,4) and (5,14). The perpendicular bisector is the line through this point, with a gradient of $-\frac{1}{m}$, where m is the gradient of the line joining the points (1,4) and (5,14).

The exercises in the textbook contain a number of such questions, which apply the ideas of this lesson to problems in Geometry.

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Suggested Answers to Activities

Activity 1

(i) 10, (4, 7)

(ii) $\sqrt{34}$, (3.5, 7.5)

(iii) $5\sqrt{2}$, (0.5, 1.5)

(iv) $\sqrt{157}$, (1.5, –9)

Activity 2

(i) $(0, \frac{1}{2})$, $(\frac{1}{4}, 0)$, gradient = –2

(ii) $(0, \frac{5}{4})$, $(\frac{5}{3}, 0)$, gradient = $-\frac{3}{4}$

(iii) $(0,\frac{3}{2})$, (3, 0), gradient = $-\frac{1}{2}$

Activity 3

parallel: (b) & (c)

perpendicular: (a) & (e)

Activity 4

(i) $y = \frac{1}{2}x - \frac{1}{2}$

(ii) $y = -\frac{3}{5}x - \frac{2}{5}$

(iii) $y = 2x - 10$

(iv) $y = 1 - x$

(v) $y = 4x - 4$

Activity 5

(i) (0,5)

(ii) (–2,10)

(iii) (2,–1)

(iv) (–3,2)