Lesson Three

Probability

Aims

The aims of this lesson are to enable you to

- calculate and understand probability
- apply the laws of probability in a variety of situations
- use tree diagrams and frequency tables to solve probability questions

Context

This section tackles the theory of probability. The basics were covered in Maths GCSE.

Statistics 1, pages 33-56

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Probability

Probability occurs in many areas of everyday activity. Examples of its use occur in weather forecasting, gambling, insurance, stock exchange, etc. In fact the fundamental laws of probability were developed back in the 17^{th} century when some gamblers at the time became curious about the laws of chance.

If someone says that the odds are "3-1 against" it raining tomorrow, they mean that it is three times as likely not to rain as it is to rain. On four days, each of which is typical of tomorrow, rain is expected on only one, and we talk of the **probability** of rain as $\frac{1}{4}$.

Another way of mathematically expressing this is: P(rain) = 0.25. Both 'odds' and 'probability' express a measure of uncertainty in different ways, but probability, being a fraction, fits in better with the mathematical theory that is being developed. Odds tend to be used by bookmakers.

The Definition of Probability

If an event **E** occurs successfully on s occasions out of a total of t, the probability of success of event **E**, which can be written P(E), is given by:

$$P(E) = \frac{s}{t}$$

The probability of a certainty must be 1 and of an impossibility must be 0. In practice all probabilities will be values lying between these limits.

The probability of event E being unsuccessful, written

P (E') or $P(\overline{E})$ is given by

$$P(\overline{E}) = \frac{t-s}{t}$$

Note that $P(E') = 1 - \frac{s}{t}$

At this point you should read pp. 33-35 and do Ex. 3A.

Probability Theorems

You will now be introduced to two important concepts connected with probability. These will be used when you have to combine two or more probabilities together. Usually it is a question of whether to add or multiply probabilities together, but you need a clear understanding of how and why these rules work.

Mutually Exclusive Events

Two events are mutually exclusive if they have nothing in common, that is, there is no overlap. In mathematical language this is defined as:

If two mutually exclusive events E_1 and E_2 have probabilities of success $P(E_1)$ and $P(E_2)$ respectively then:

$$P(E_1 \textbf{ or } E_2) = P(E_1) + P(E_2)$$

You may also see this written as $P(E_1 \cup E_2)$ or $p(E_1 \cup E_2)$ where $\cup$ is the symbol for "or".

Thus the probability of throwing a 4 or a 5 with a single die can be calculated as $\frac{1}{3}$ either directly, or by using the theorem as $\frac{1}{6} + \frac{1}{6}$.

Note that if $E_1 = E$ and $E_2 = E'$ we obtain:

$$P(E) + P(E') = 1$$

confirming the previously obtained result:

$$P(E') = 1 - P(E)$$

Independent Events

This effectively means that the probability of one event happening does not affect the probability of another event happening. The precise mathematical definition is:

If two independent events E_1 and E_2 have probabilities of success $P(E_1)$ and $P(E_2)$ respectively then:

$$P(E_1 \textbf{ and } E_2) = P(E_1) \times P(E_2)$$

You may also see this written as $P(E_1 \cap E_2)$ or $p(E_1 \cap E_2)$ where $\cap$ is the symbol for "and"

Thus the probability of throwing a 4 with a single die is $\frac{1}{6}$. The probability of throwing two 4s with two dice is $\frac{1}{36}$. This can be calculated directly from the definition of probability or by the theorem as $\frac{1}{6} \times \frac{1}{6}$.

Example (a)

Find the probability of throwing 2 dice and obtaining a total of less than 6.

Letting (a, b) represent a score of **a** on the first die and **b** on the second die, the totals which meets the requirements of the question can be obtained in the following ways:

(1, 1) (1, 2) (1, 3) (1, 4) (2, 1) (2, 2) (2, 3) (3, 1) (3, 2) (4, 1)

These are all mutually exclusive possibilities, each with a probability of $\frac{1}{36}$. Required probability = $\frac{10}{36} = \frac{5}{18}$

Example (b)

A committee of 3 men and 2 women is to be chosen from a total of 5 men and 3 women, including a married couple. Find the probability of:

(a) both members of the couple serving on the committee.

(b) the husband only serving.

(c) the wife only serving.

(d) neither serving.

The probability of the husband serving can be written $P(h)$ and of the wife serving can be written $P(w)$. The probabilities are:

(a) $P(h \text{ and } w) = P(h) \times P(w) = \frac{3}{5} \times \frac{2}{3} = \frac{2}{5}$

(b) $\quad P(h \text{ and } \overline{w}) = P(h) \times P(\overline{w}) \quad = \frac{3}{5} \times \frac{1}{3} = \frac{1}{5}$

(c) $\quad P(\overline{h} \text{ and } w) = P(\overline{h}) \times P(w) \quad = \frac{2}{5} \times \frac{2}{3} = \frac{4}{15}$

(d) $\quad P(\overline{h} \text{ and } \overline{w}) = P(\overline{h}) \times P(\overline{w}) \quad = \frac{2}{5} \times \frac{1}{3} = \frac{2}{15}$

Note that the total of probabilities is 1, as would be expected since all possibilities are covered. Therefore, (d) could have been calculated, using this fact, as:

$$1 - \frac{4}{15} - \frac{1}{5} - \frac{2}{5} = \frac{2}{15}$$

taking the results of (a), (b) and(c).

Example (c)

If three coins are tossed together, what is the probability that they will fall alike?

P(3 alike) = P((H and H and H) or (T and T and T))

$$= \left(P(H) \times P(H) \times P(H)\right) + \left(P(T) \times P(T) \times P(T)\right)$$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = 0.25$$

Now do Exercises 3B and 3C.

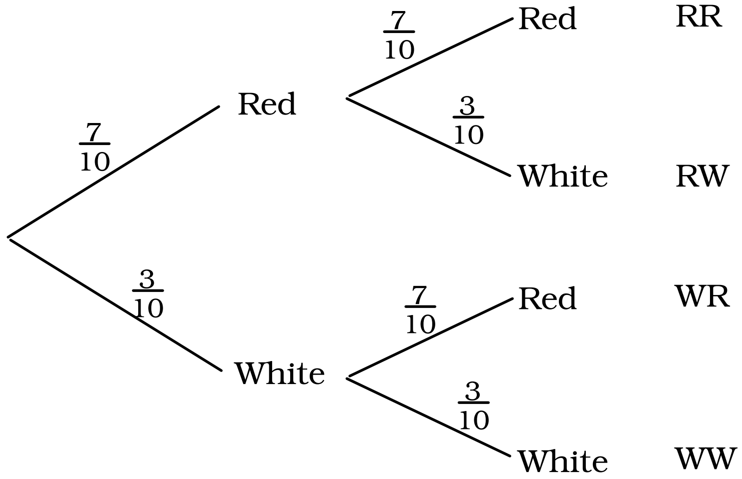
Tree Diagrams

A tree diagram shows the possible outcomes and the probabilities of each outcome happening along its branches.

Example

Consider this situation. A bag contains 7 red balls and 3 white balls. One is selected at random and then returned to the bag. Another is then selected. We want to list the various outcomes and find their probability.

This diagram shows this:



The various outcomes are:

RR RW WR WW

To find the probability of each of these, you need to multiply the probabilities on each branch.

e.g. the probability of getting two red balls is:

$$\frac{7}{10} \times \frac{7}{10} = \frac{49}{100} \quad \text{or } 0.49$$

To find the probability of getting a ball of each colour (in either order) you need to find the probabilities at the end of two branches and add them:

$$\frac{7}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{7}{10} = \frac{42}{100} \quad \text{or } 0.42$$

So the two 'rules' to remember are:

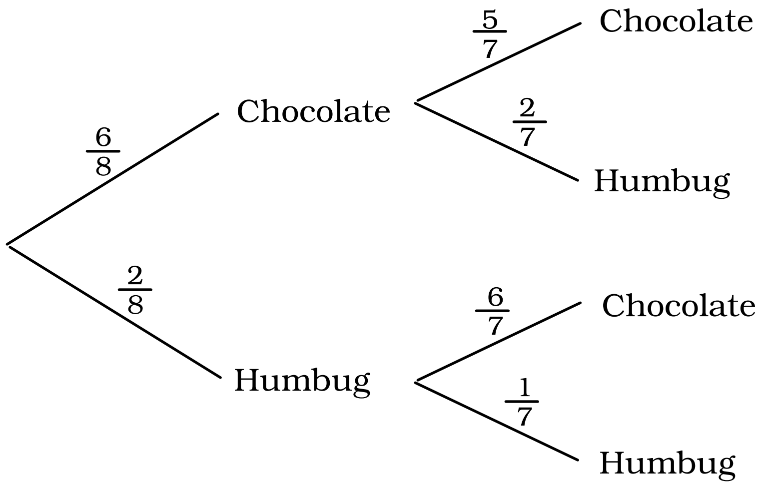
- The probability of any event represented by a single branch of the tree is the product of the probabilities along the branch.
- The probability of any event which involves two or more branches is the sum of the probabilities of each branch.

Example

The probabilities on the branches are not always the same each time an event occurs. Consider this example:

A bag contains 8 sweets, 6 chocolates and 2 humbugs, each wrapped similarly so that you cannot tell which you are selecting. This time a sweet is taken from the bag and eaten, then another one taken and eaten, so they are not replaced.

The tree diagram looks like this:



The denominator of the fractions on the second branches goes down to 7 because there are now only 7 sweets in the bag.

The probabilities can be worked out as in the previous example.

If you would like further examples of tree diagrams look at p.41. Otherwise do Exercise 3D.

Joint Probability and the General Law of Addition

Joint probability is the probability that measures the likelihood that two or more events will happen concurrently.

The key question to ask is whether the two events are mutually exclusive. For instance, if we drew one card from a deck, what is the probability that it is a jack *or* a spade? The chances of a jack are $\frac{1}{13}$ and the chances of a spade are $\frac{1}{4}$, so the probability of getting one or the other would appear to be $\frac{1}{13} + \frac{1}{4} = \frac{17}{52}$

But apart from the 13 spades, there are only three other jacks, so it is easy to see that the true answer is $\frac{16}{52}$. The problem is that the card might be *both* (i.e. the Jack of spades) and we need to take account of it in our overall calculations. To take account of this, where events are *not* mutually exclusive, we need a **general rule of addition**, as follows:

P(A or B) = P(A) + P(B) – P(A and B)

In this case we get $\frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{16}{52}$

We will look at the consequences of this rule in more detail in the next lesson.

Conditional Probability

If A and B are two events, then the conditional probability of event B, given A has occurred, is denoted by P(B|A). "|" means "given that", so this means the probability of B, given that A has already occurred.

The conditiona1 probability only arises where one event is dependent on another event having occurred. If two events, A and B, are independent then P(B|A) = P(B).

Example

What are the chances of drawing two aces in a row from a pack of cards, once you've already picked one? There are two possible answers to this and the question actually needs to be clarified.

Possibility 1: Dependent

What is the probability of drawing a second ace from a pack of cards *if the first ace is not replaced before the second draw is made*?

P (drawing the first ace) = $\frac{4}{52}$ = 0.077

Now suppose that the first ace has been drawn. Since the first card is not replaced, the second event is dependent upon the first event. There are now only 51 possible outcomes, of which 3 would produce the desired event:

i.e. an ace is drawn.

$\therefore$ P(drawing a second ace) = $\frac{3}{51}$ = 0.059

Possibility 2: Independent

What is the probability of drawing a second ace from a pack of cards *if the first ace is replaced before the second draw is made*?

Since the first card is replaced, the two events are independent.

$\therefore$ P(drawing a second ace) = $\frac{4}{52}$ again = 0.077

Multiplication Law

The probability that both event A and event B occur together is the product of the probability that event A occurs and the probability that event B occurs given that event A has already occurred.

i.e. $P(A \cap B) = P(A) \times P(B|A)$

If events A and B are independent then:

$$P(B|A) = P(B) \text{ and } P(A \cap B) = P(A) \times P(B)$$

Example

What is the probability of drawing two aces from a pack of cards, if the first card is not replaced before the second card is drawn?

P(drawing first ace) = $\frac{4}{52}$ = 0.077

P(drawing a second ace) = $\frac{3}{51}$ = 0.059

P(Drawing two aces without replacement)

$$= \frac{4}{52} \times \frac{3}{51} = 0.0045$$

$$= \frac{1}{221} = 0.0045$$

Example (a)

What is the probability of drawing two aces from a pack of cards if the first card is replaced before the second card is drawn?

P(drawing first ace) $= \frac{4}{52} = 0.077$

P(drawing second ace) $= \frac{4}{52} = 0\ 077$

$\therefore$ P(drawing two aces) $= \frac{4}{52} \times \frac{4}{52}$

$= \frac{1}{169} = 0.0059$

Example (b)

A bag contains three white and four black balls. A ball is drawn at random and not replaced. A second ball is drawn at random. What is the probability that the balls are different colours?

There are two possibilities:

Either:

a white ball is drawn, then a black ball

or:

a black ball is drawn, then a white ball

These two events are mutually exclusive (since both a white and black ball cannot be drawn at the same time).

$\therefore$ P(balls different colours) = P(A) + P(B)

Now consider P(A).

P(drawing a white ball) $= \frac{3}{7} = 0.428$

P(drawing a black after a white ball)

$= \frac{4}{6} = 0.667$

P(A) $= \frac{3}{7} \times \frac{4}{6}$

$$= \frac{2}{7} = 0.285$$

Now consider P(B):

P(drawing a black ball) $= \frac{4}{7} = 0.571$

P(drawing a white ball after a black ball)

$$= \frac{3}{6} = 0.5$$

$$\therefore P(B) = \frac{4}{7} \times \frac{3}{6}$$

$$= \frac{2}{7} = 0.285$$

P(balls different colours) $= \frac{4}{7} = 0.57$

Example (c)

If one card is drawn at random from a full pack of 52 playing cards, the probability that it is red is $\frac{26}{52}$ or 0.5. If the second card is selected at random from the pack (without replacing the first), the probability that it is red depends on the colour of the first card drawn. There are only 51 cards left that could be selected as the second card, all of them having an equal chance. If the first had been black, there are 26 red cards available, and the probability that the second card is red is therefore $\frac{26}{51}$ or 0.510 (i.e. conditional upon the colour of the first card.)

Similarly, if the first card is red, the probability that the second is also red is 0.49. Using the multiplication law, and conditional probabilities discussed above, the probability that two cards, taken randomly from a full pack (52 cards), will both be red, is given by

P(first card red) $\times$ P(second card also red)

$$= \frac{26}{52} \times \frac{25}{51} = \frac{25}{102} = 0.245$$

The concepts in this section are not easy and can lead to some tricky problems. The examiners for S1 like to test these ideas using frequency tables. The following example is worthy of serious study:

Frequency Table Example

Football supporters in East Anglia support either Norwich City or Ipswich Town. A random sample of 200 supporters produced an age profile as shown below:

	Young (Y)	Middle-aged	Old
Norwich(N)	26	47	27
Ipswich	26	56	18

A supporter is randomly chosen. The event Y means the person is Young and the event N means the person supports Norwich City.

1. Find probability of:

   a) N

   b) Y'

   c) $N \cup Y$

   d) $N' \cap Y$

   e) $N|Y$

2. Give two events which are

   i. independent

   ii. mutually exclusive

   Justify your answer in each case.

You should begin questions such as this by totalling the rows and columns. This gives:

	Young (Y)	Middle-aged	Old	Total
Norwich (N)	26	47	27	100
Ipswich	26	56	18	100
Total	52	103	45	200

(a) $P(N) = \dfrac{100}{200} = \dfrac{1}{2}$

(b) $P(Y') = \dfrac{47+27+56+18}{200} = \dfrac{148}{200} = \dfrac{74}{100}$

(c) $P\ (N \cup Y) = \dfrac{47+27+26+26}{200} = \dfrac{126}{200} = \dfrac{63}{100}$

(d) $P\ (N' \cap Y) = \dfrac{26}{200} = \dfrac{13}{100}$

(e) $P\ (N \mid Y) = \dfrac{26}{52} = \dfrac{1}{2}$

Since $P(N) = P\ (N \mid Y) = \dfrac{1}{2}$ then N and Y are independent events

Two events which are mutually exclusive are Young and Old as a person is either classified as one or the other and cannot be both. (There are other possibilities here as well.)

Read pp. 44-50 and do Exercise 3E.

In Conclusion

Probability starts fairly easily but, by the end of this lesson, it's getting a bit more tricky!

Exercise 3F and the Mixed Exercise are best left for revision as they are quite tough. Return to them every now and again to keep Probability ticking over.

Self-Assessment Test Three

1. A bag contains 3 white, 4 blue and 5 red balls. If two balls are removed, find the probabilities of:

(a) both being the same colour.

(b) neither being red.

(c) a blue and a red ball are drawn.

2. As part of a quality check, items coming off a production line are examined by 2 inspectors. About 5% of defective items escape detection by the first inspector and 80% of these are detected by the second inspector. What are the chances that a defective item will get by both inspectors?

3. The census for a particular rural district revealed that 250 households lived in the area. The following information concerning the number of children in the households together with the workplace of the head-of-household was collected.

Number of children	Workplace of head-of-household		
	Local	Nearby Town	Unemployed
0	40	5	0
1	50	15	2
2	60	20	1
3	30	5	1
4 or more	20	0	1

If one household is selected at random what is the probability that the household will have:

(a) at least 2 children?
(b) at most 2 children?
(c) exactly 2 children?
(d) a head-of-household working in a nearby town?
(e) a head-of-household working locally with not more than 1 child?
(f) an unemployed head-of-household with no children?