Lesson Five

Polynomials

Aims

The aims of this lesson are to enable you to

- multiply and divide polynomials
- factorise polynomials
- use the Factor and Remainder theorems
- sketch the graphs of cubic functions

Context

Polynomials appear throughout Mathematics. This lesson covers some of the tools that will be useful in later work.

References are made throughout this lesson to the Heinemann textbook, which provides further worked examples and exercises.

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Introduction

The following are all examples of polynomials:

(a) $2x + 7$

(b) $3x^2 - 4x + 5$

(c) $4x^2 - 7$

(d) $1 + 5x - 6x^3$

(e) $3x^5 + 5x^4 - 2x^2 + x$

Example (a) would usually be referred to as a **linear** expression (in x), whilst (b) and (c) are **quadratic** expressions (or just 'quadratics'). Example (d) would be described as a **cubic**, whilst (e) could be described as a 5th **order** (or 5th **degree**) polynomial (although the terms 'quartic' and 'quintic' are sometimes used for 4th and 5th order polynomials, respectively).

The multiples of the powers of x are termed '**coefficients**'. Thus the coefficient of x^4 in (e) is 5.

If there is a term which is a number only (i.e. does not involve x), then this is referred to as the **constant** term. Thus the constant term in (d) is 1.

It is possible to have a polynomial involving some other letter, but x is by far the most commonly used.

Polynomials can be added or subtracted, by grouping together the same powers of x.

Example 1

$$(2x + 7) + (3x^2 - 4x + 5) = 3x^2 - 2x + 12$$

Example 2

$$(2x + 7) - (3x^2 - 4x + 5) = 2 + 6x - 3x^2$$

Polynomials tend to be written in descending powers of x for most purposes, but note how the answer in Example 2 has been written in ascending powers in order to avoid starting with $-3x^2$.

Note: Polynomials can only be added or subtracted if they involve the same letter (or 'variable').

Practice at adding and subtracting polynomials (if needed) can be obtained in Exercise 5A of the textbook.

Multiplying Polynomials

You will be familiar with the multiplication of linear expressions

e.g. $(x + 2)(2x - 3) = 2x^2 - 3x + 4x - 6 = 2x^2 + x - 6$

The same process can be applied to higher order polynomials:

Example 3

$(2x + 7)(3x^2 - 4x + 5) = 6x^3 - 8x^2 + 10x + 21x^2 - 28x + 35$

$= 6x^3 + 13x^2 - 18x + 35$

Example 4

$(x - 1)(x + 2)(x + 1)$

Method 1: $(x^2 + x - 2)(x + 1) = x^3 + x^2 + x^2 + x - 2x - 2$
$= x^3 + 2x^2 - x - 2$

Method 2: Spotting the 'difference of two squares':

$(x - 1)(x + 1) = x^2 - 1,$

we have $(x^2 - 1)(x + 2) = x^3 + 2x^2 - x - 2$

Note: As for ordinary multiplication of numbers, the order of the factors isn't important.

Activity 1

Multiply together the following polynomials, and simplify the result.

(i) $x + 1$ and $2x^2 - x + 3$

(ii) $x^2 - 2$ and $x^2 + x - 1$

(iii) $x - 1$ and $x^2 + x + 1$

Further practice at multiplying polynomials can be obtained in Exercise 5B of the textbook.

Division of Polynomials by Linear Factors

Example 5

Suppose that we are given a polynomial such as $2x^3 + 3x^2 - 2x + 4$ and wish to write it in the form $(x-2)Q(x) + R$

where $Q(x)$ is a quadratic expression and R is a constant.

In other words, we are trying to divide the polynomial by $(x - 2)$, and there may or may not be a remainder R.

$Q(x)$ is termed the quotient.

This can be done as follows:

The x^2 term in $Q(x)$ must be $2x^2$, as this is the only way that we will obtain the required $2x^3$.

We then have $(x-2)(2x^2 + \ldots)$, and expanding what we have so far gives:

$2x^3 - 4x^2$

However, we want $+3x^2$ instead of $-4x^2$, and therefore we need to be adding $7x^2$. The only way that we will obtain this is by making the x term $7x$.

We then have $(x-2)(2x^2 + 7x + \ldots)$, and expanding this gives:

$2x^3 - 4x^2 + 7x^2 - 14x$

We want $-2x$ instead of $-14x$, and so we need another $12x$, which will come from making the constant term in the quadratic equal to 12.

We then have $(x-2)(2x^2 + 7x + 12)$, which expands to give:

$2x^3 - 4x^2 + 7x^2 - 14x + 12x - 24 = 2x^3 + 3x^2 - 2x + 4 - 28$

So $2x^3 + 3x^2 - 2x + 4 = (x-2)(2x^2 + 7x + 12) + 28$

and we have a remainder of 28.

Another way of expressing this result is to say:

$2x^3 + 3x^2 - 2x + 4$ divided by $x - 2 = \frac{2x^3 + 3x^2 - 2x + 4}{x-2}$

$= 2x^2 + 7x + 12 + \frac{28}{x-2}$

Note: It is also possible to carry out this division using the traditional long division, adapted to polynomials. However, the above method is quicker.

Example 6

Factorise $x^3 - 6x^2 - 9x + 14$ fully, given that $(x - 1)$ is a factor.

$x^3 - 6x^2 - 9x + 14 = (x - 1)(x^2 + \ldots)$

So far we have $x^3 - x^2$ and we therefore need to make an adjustment of $-5x^2$ (in order to obtain the desired $-6x^2$).

The next term in the quadratic therefore has to be $-5x$, to give

$(x - 1)(x^2 - 5x + \ldots) = x^3 - x^2 - 5x^2 + 5x + \ldots$

The next adjustment required is $-14x$, so that the last term in the quadratic has to be -14, and we have

$x^3 - 6x^2 - 9x + 14 = (x - 1)(x^2 - 5x - 14)$

Note that this is self–checking, as the constant term of 14 is obtained when we expand the brackets.

Finally we need to see if $x^2 - 5x - 14$ can be factorised, and we thus obtain

$x^3 - 6x^2 - 9x + 14 = (x - 1)(x + 2)(x - 7)$ as the complete factorisation.

Activity 2	Write each of the following polynomials in the form $(x - a)Q(x) + R$, where R is a constant (which may be zero):
	(i) $5x^3 + 2x^2 - x + 7$ where a = 2 (ii) $2x^3 - 11x^2 + 22x - 61$ where a = 4 (iii) $4x^3 + 11x^2 - 2x + 3$ where a = –3

Example 7

Divide $x + 7$ by $x + 8$

All we need to do is write $x + 7 = (x + 8) - 1$, and divide both sides by $x + 8$, to give:

$$\frac{x+7}{x+8} = 1 - \frac{1}{x+8}$$

Example 8

Divide $5x^3 + 5x^2 - 6x - 4$ by $x + 1$

This example will be carried out by another approach, which, although not quicker in this case, will prove to be valuable in other situations later in the course.

Suppose that $5x^3 + 5x^2 - 6x - 4 = (x + 1)(ax^2 + bx + c) + d$

This has to be true for all values of x, and the only way in which this can happen is if the coefficient of x^3 on the left–hand side (5) equals the coefficient of x^3 on the right–hand side (a). Thus $a = 5$.

So far, this is only really what we have been doing all along. But we can apply this idea to the other powers of x as well:

x^2: $5 = a + b$ (we look to see which terms involve x^2 when the right–hand side is expanded)

As we know that $a = 5$, it follows that $b = 0$.

x: $-6 = b + c$, which means that $c = -6$

constant term (or x^0): $-4 = c + d$, so that $d = -4 - (-6) = 2$

So we have $5x^3 + 5x^2 - 6x - 4 = (x + 1)(5x^2 - 6) + 2$ (A)

and this could be checked by expanding the right–hand side.

This approach is known as the method of **equating coefficients**.

As we are asked to divide $5x^3 + 5x^2 - 6x - 4$ by $x + 1$, the final answer is

$5x^2 - 6 + \frac{2}{x+1}$ (we are dividing both sides of (A) by $x + 1$).

Activity 3	Carry out each of the following divisions:

(i) $(x + 7) \div (x - 2)$

(ii) $(2x + 5) \div (x - 7)$

(iii) $(x^3 + 4x^2 - x - 4) \div (x + 4)$

Further practice at dividing polynomials by linear factors can be obtained in Exercise 6D of the textbook.

The Factor Theorem

In Example 6 we were able to factorise $x^3 - 6x^2 - 9x + 14$ fully to give

$x^3 - 6x^2 - 9x + 14 = (x - 1)(x + 2)(x - 7)$ (B)

Notice what happens when we put $x = 1$. The right-hand side equals zero, because $(x - 1)$ is a factor.

Because (B) is true for all values of x, the left–hand side must also equal zero when $x = 1$.

If we use the notation $f(x) = x^3 - 6x^2 - 9x + 14$, then $f(1) = 0$.

Similarly $f(7) = 0$ and $f(-2) = 0$. Note that the values of x that are chosen are the ones that make the linear factors zero.

In general, if $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$.

The opposite is also true: if $f(a) = 0$, then $(x - a)$ must be a factor of $f(x)$.

This is the **Factor Theorem**.

Example 9

Factorise $f(x) = x^3 + 3x^2 - 13x - 15$

There might not seem much to go on here, but the Factor Theorem can be used, by trying some small values of x.

First of all, suppose that $(x - 1)$ is a factor. We are hoping that $f(1) = 0$.

Now $f(1) = 1 + 3 - 13 - 15 = - 24$. So that doesn't work.

The next simplest value to try is -1:

$f(-1) = -1 + 3 + 13 - 15 = 0$

So $(x + 1)$ is a factor, and we can write:
$x^3 + 3x^2 - 13x - 15 = (x + 1)(x^2 + \dots)$

It is possible to continue trying out other values of x, but they may not turn out to be small numbers. In any case, we can find the quadratic on the right–hand side by the usual method, and then factorise that quadratic.

So we get $(x + 1)(x^2 + 2x - 15)$ [you might like to check this]

which factorises to give $(x + 1)(x - 3)(x + 5)$.

Note: We can tell from the constant term in $f(x)$ [-15] that if $(x - a)$ or $(x + a)$ is to be a factor of $f(x)$, then a must be a factor of 15. So there would be no point in trying the value 2, for example.

Activity 4

Factorise fully the following cubics, and hence solve the equations $f(x) = 0$

(i) $f(x) = x^3 - 2x^2 - 11x + 12$

(ii) $f(x) = x^3 - x^2 - 4x + 4$

(iii) $f(x) = 2x^3 + 9x^2 - 33x + 14$

(iv) $f(x) = x^3 + 1$

Further practice at using the Factor Theorem can be obtained in Exercise 6A of the textbook.

The Remainder Theorem

In the previous section we considered $f(x) = x^3 - 6x^2 - 9x + 14$, which could be factorised fully as $(x-1)(x+2)(x-7)$.

Suppose now that we change f(x) slightly,

and let $g(x) = x^3 - 6x^2 - 9x + 15$.

Then we know that $g(x) = (x-1)(x+2)(x-7) + 1$

The remainder when g(x) is divided by $x-1$, $x+2$ or $x-7$ is 1.

And $g(1) = 0 + 1 = 1$, since the factor $x-1$ becomes 0 when $x = 1$.

Similarly, $g(-2)$ and $g(7)$ also equal 1.

In general, if the polynomial g(x) is divided by $(x - a)$, then the remainder will be $g(a)$.

This is the **Remainder Theorem**.

The Factor Theorem is in fact the special case of the Remainder Theorem where the remainder is zero.

Example 10

Find the remainder when $2x^3 + 3x^2 - 2x + 4$ is divided by $(x-2)$

Let $f(x) = 2x^3 + 3x^2 - 2x + 4$. Then the remainder when f(x) is divided by $(x-2)$ is

$f(2) = 16 + 12 - 4 + 4 = 28$

f(x) in fact appeared in Example 5, where we saw that

$$2x^3 + 3x^2 - 2x + 4 = (x-2)(2x^2 + 7x + 12) + 28$$

Important point: If a question only asks for the remainder (and not the quotient [$2x^2 + 7x + 12$ in this case]) when a polynomial is divided by a factor of the form $(x-a)$ [or $(x+a)$], then there is no need to actually carry out the division: the Remainder Theorem can be used instead. Similarly, if we only need to decide whether $(x-a)$ is a factor of a polynomial, then the Factor Theorem is all that is needed.

Activity 5	Factors and Remainders

	(i) Is $(x - 1)$ a factor of $8x^3 - 2x^2 - 3x - 3$? If not, give the remainder. (ii) Is $(x + 2)$ a factor of $2x^3 + 5x^2 + x - 5$? If not, give the remainder. (iii) If $(x + 1)$ is a factor of $kx^3 + 4x^2 + 6$, what is the value of k? (iv) Is $(x - 3)$ a factor of $16x^2 - 53x + 15$? If not, give the remainder. (v) The remainder when $3x^3 - kx^2 - 3x + 6$ is divided by $(x - 2)$ is 4. Find k.

Further practice at using the Remainder Theorem can be obtained in Exercise 6E of the textbook.

Sketching Graphs of Cubic Functions

First of all, note the basic shape of the simplest possible cubic function: $y = x^3$

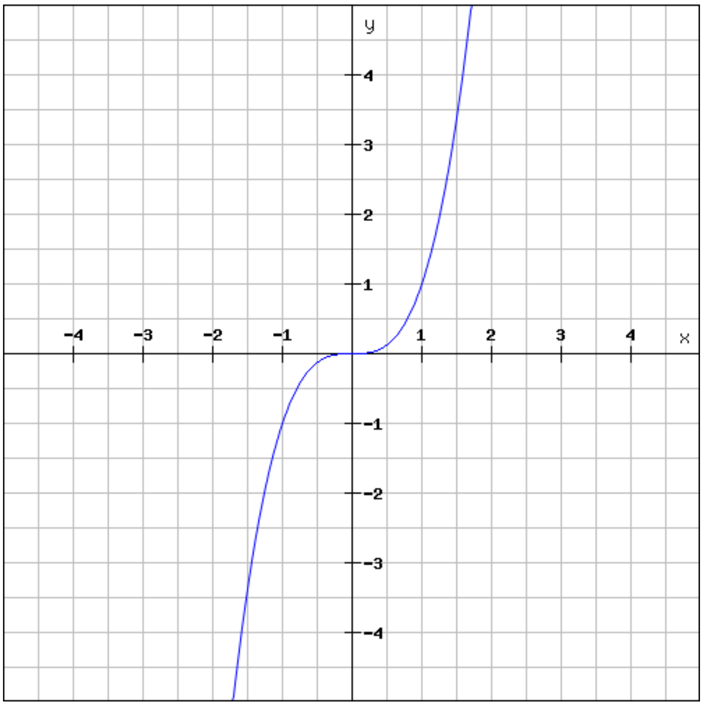


Figure 1: $y = x^3$

If a cubic function has been factorised, then we know where its graph crosses the x-axis.

Example 11

Sketch $f(x) = x^3 + 3x^2 - 13x - 15$

In Example 9 we saw that $f(x)$ could be factorised as $(x + 1)(x - 3)(x + 5)$.

This means that the equation $f(x) = 0$ has the solutions -5, -1 and 3, and these are the x-coordinates of the points where the graph of $f(x)$ crosses the x-axis.

The graph crosses the y-axis when $x = 0$, i.e. when $y = -15$.

When x is large and positive, $f(x)$ will be large and positive. Provided that the coefficient of x^3 is positive, this will be true whatever the coefficients of the other powers of x, when x is large enough (because the x^3 term outweighs the other terms).

Also, when x is large and negative, $f(x)$ will be large and negative.

When asked to sketch a curve, these are the features that are relevant:

- (roughly) where the curve 'starts' and 'ends'
- where the curve crosses the x-axis
- where the curve crosses the y-axis

For $f(x)$ above, a sketch might look as follows:

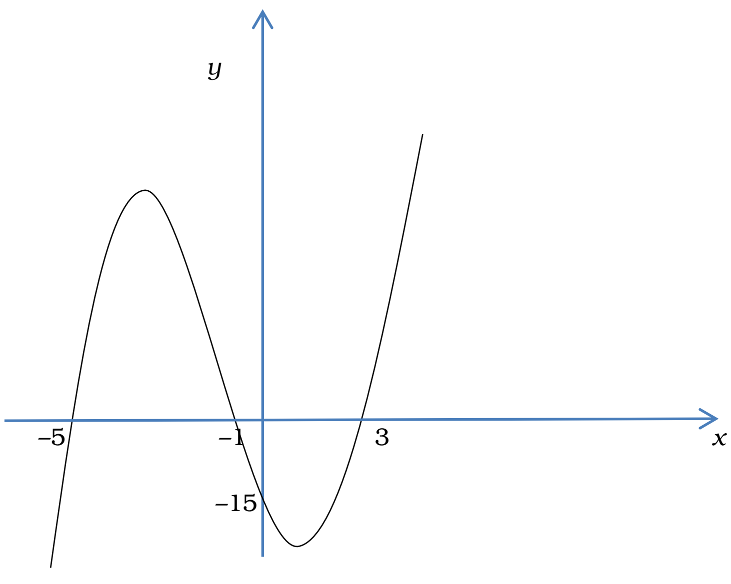


Figure 2

Note the following:

The points where the curve crosses the x and y–axes should be labelled.

The scale on the y-axis does not need to be the same as the scale on the x-axis.

The precise location of the 'turning points' of the curve (between $x = -5$ and -1, and between $x = -1$ and 3) does not need to be found. It is sufficient to make the curve as smooth as possible (i.e. avoiding sharp changes in direction).

Example 12

Sketch $f(x) = 2(x + 1)^2(2 - x)$

First of all, the curve will 'start' at the top and 'end' at the bottom, since there will be a $-2x^3$ term (consider what happens when x is large and negative, and when x is large and positive).

Then there is a **repeated root** at $x = -1$. This means that the curve just touches the x-axis at this point (compare with $y = x^2$, where the curve touches the x-axis at $x = 0$).

By expanding $f(x)$, we can see that the y–intercept is at $y = 4$.

So a sketch might look as follows:

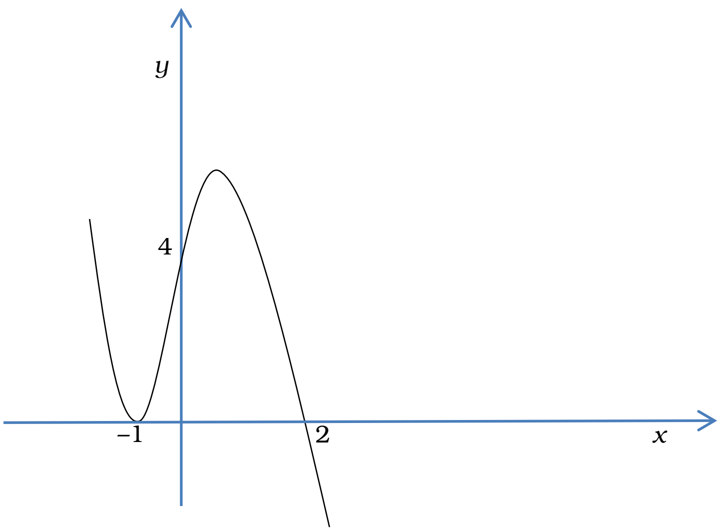


Figure 3

Example 13

Sketch $f(x) = (x-1)(x^2 + 2x + 5)$

Note first of all that the equation $x^2 + 2x + 5 = 0$ has no roots (since $b^2 - 4ac < 0$). So the curve only crosses the x-axis at $x = 1$.

The y–intercept is at $y = -5$.

Here there is not as much to go on, but a sketch should look roughly as follows:

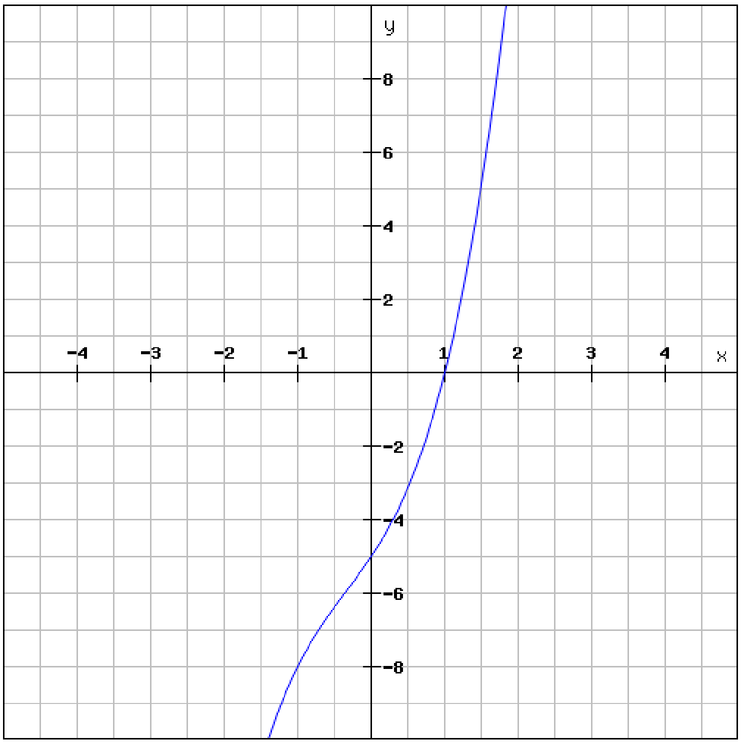


Figure 4

Example 14

Sketch $f(x) = 2(x + 2)^3$

Finally, if the equation $f(x) = 0$ has 3 repeated roots, the curve will have a similar pattern to that of $y = x^3$, and a sketch should look roughly as follows:

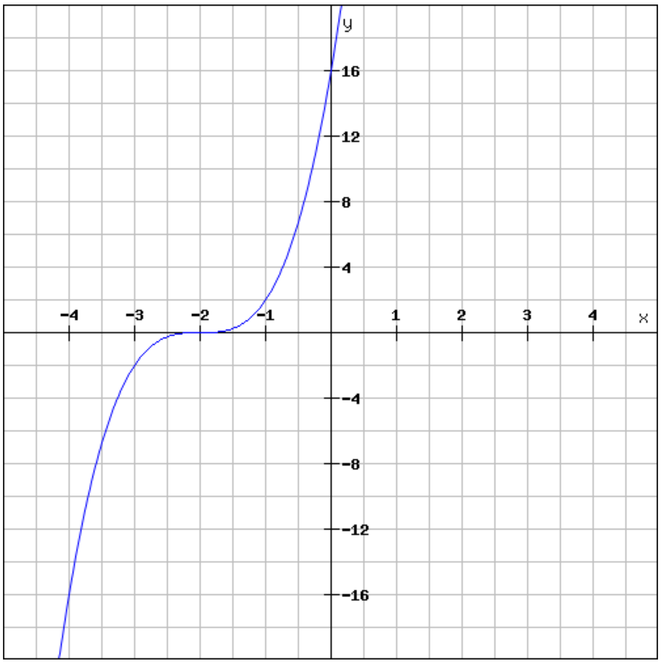


Figure 5

Activity 6	Factorise the following cubic functions and hence sketch their curves:
	(i) $2x^3 - 3x^2 - 8x - 3$ (ii) $9 - 21x + 15x^2 - 3x^3$ (iii) $x^3 - 3x^2 + 4x$

Further practice at sketching cubic functions can be obtained in Exercise 6C of the textbook.

Suggested Answers to Activities

Activity 1

(i) $2x^3 + x^2 + 2x + 3$

(ii) $x^4 + x^3 - 3x^2 - 2x + 2$

(iii) $x^3 - 1$

Activity 2

(i) $(x-2)(5x^2+12x+23)+53$
(ii) $(x-4)(2x^2-3x+10)-21$
(iii) $(x+3)(4x^2-x+1)$

Activity 3

(i) $1+\frac{9}{x-2}$
(ii) $2+\frac{19}{x-7}$
(iii) $x^2-1=(x-1)(x+1)$

Activity 4

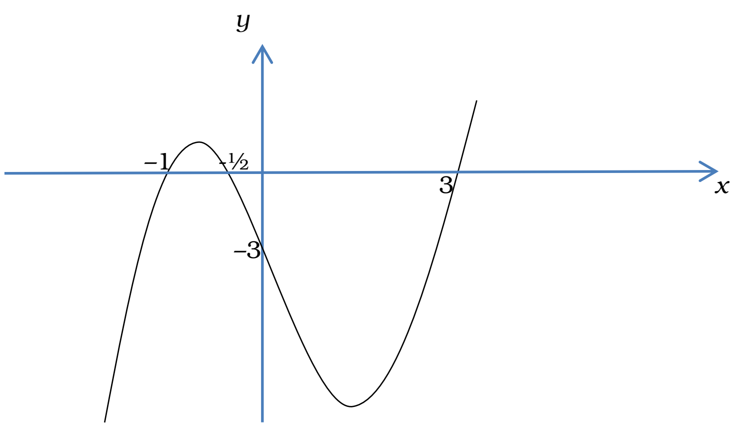
(i) $(x-1)(x+3)(x-4)$; $x=1$, -3 or 4
(ii) $(x-1)(x+2)(x-2)$; $x=1$, -2 or 2
(iii) $(x-2)(2x-1)(x+7)$; $x=2$, $\frac{1}{2}$ or -7
(iv) $(x+1)(x^2-x+1)$; $x=-1$

Activity 5

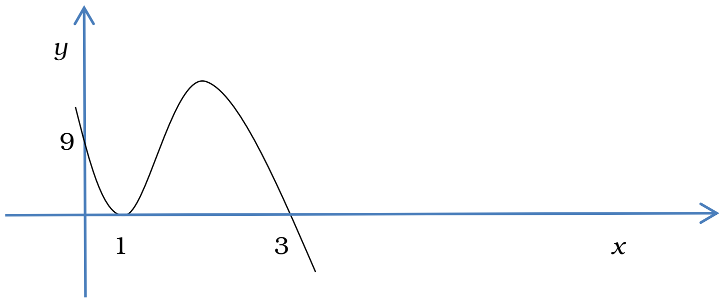
(i) Yes
(ii) No; remainder = –3
(iii) $k=10$
(iv) Yes
(v) $k=5$

Activity 6

(i) $(x+1)(2x+1)(x-3)$



(ii) $3(\boldsymbol{x}-\mathbf{1})^2(3-x)$



(iii) $x(x^2-3x+4)$