Lesson Nine

Linear Functions

Aims

The aim of this lesson is to enable you to:

- find the gradient of a straight line from a graph plot the graph of part of a linear function
- recognize parallel lines
- find the equation of a straight line from a graph
- find the equation of a straight line algebraically relate this to the rate of change or ratio

Context

Linear functions have many important applications, They are some of the simplest rules that relate one variable to another. Here we look at the relationship between the graph of a straight line and the algebraic rules that govern them so that you are able to recognize a linear function both algebraically and graphically. This will lead on to work on other graphs relating two variables and other rules.

Oxford Open Learning

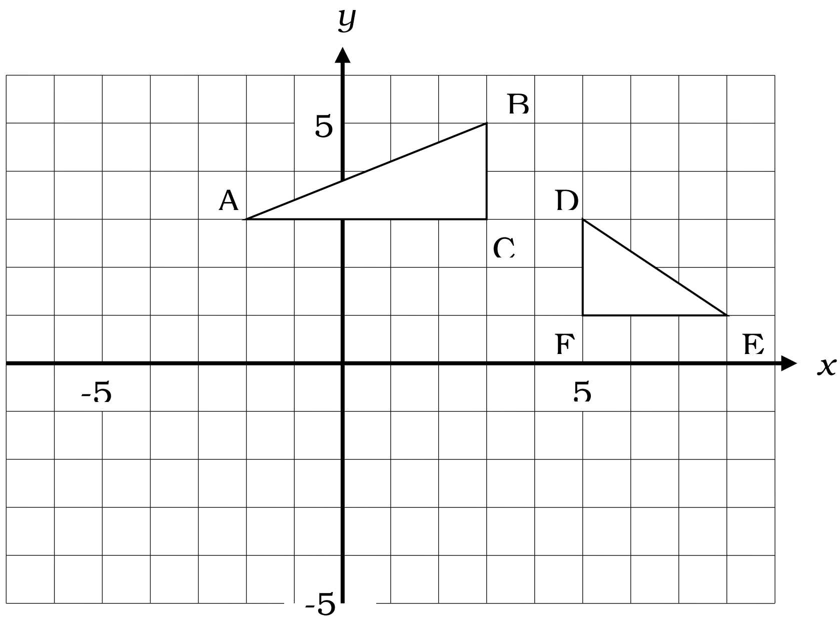
Gradient

Gradient is a measure of slope: how steep a line is. The gradient of a line segment AB is defined as the ratio of the vertical to horizontal change $\frac{change\ in\ y}{change\ in\ x}$ as we travel from A to B. (It turns out that it makes no difference if we travel from B to A: the result is the same.) We will see later that we can also think of this gradient as a rate of change: the change in y per change in x.

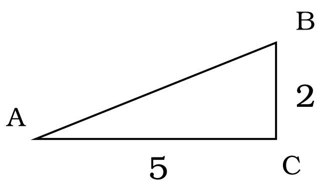
Example 1

(a) Find the gradient of the line segment AB where A and B have co-ordinates (–2, 3) and (3, 5) respectively.

(b) Find the gradient of the line segment DE where D and E have co-ordinates (5, 3) and (8, 1) respectively.

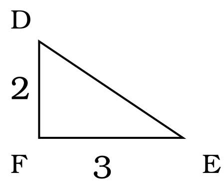


(a) The points A and B area plotted on the above grid. The point C is also plotted, so that ABC is a right-angled triangle.



We see from the grid that the lengths of AC and BC are 5 and 2 units respectively. AC represents the increase in x, and BC represents the increase in y as we travel from A to B. The gradient of AB is therefore $\frac{2}{5}$ or 0.4.

(b) the points D and E have been plotted on the grid together with the point F(5, 1), so that DEF is a right-angled triangle.



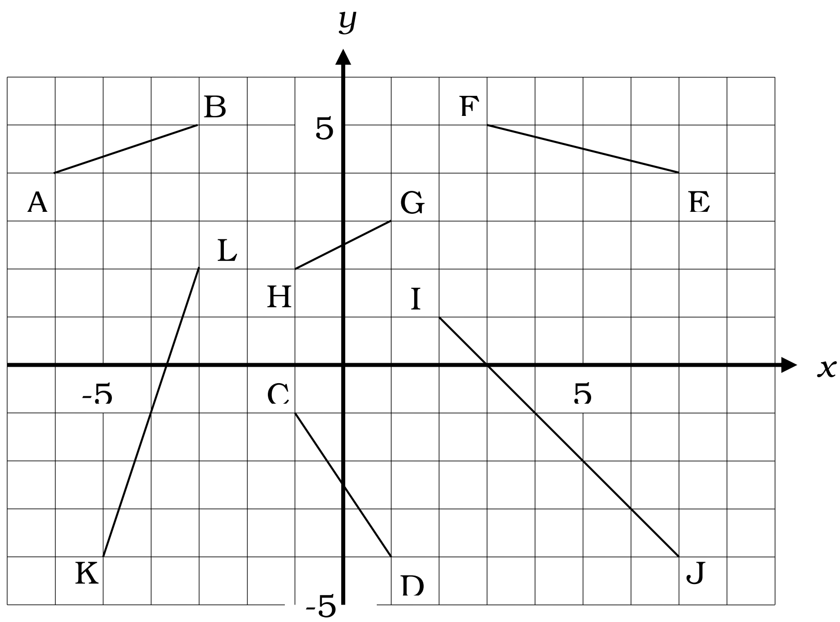
We see from the grid that the lengths of EF and DF are 3 and 2 units respectively. The increase in x is certainly 3. However, as we travel from D to E, y does not increase: it decreases. We therefore say that the 'increase in y' is negative. The gradient of DE is $\frac{-2}{3}$ or $-\frac{2}{3}$.

Positive or Negative Gradient?

Positive gradient is 'uphill' and negative gradient is 'downhill'. However, we need to be clear about the direction of travel when deciding the sign of the gradient. We must travel so that x increases. So for the points A and B, we travel from A to B, and see that we are going uphill, and the gradient is **positive**. For the points D and E, x increases as we travel from D to E. We see that we are going downhill from D to E, hence the **negative** gradient. In practice, all we do is to move left to right, and ask whether we are going uphill or downhill.

Example 2

Which of the following line segments have:
(a) a positive gradient (b) a negative gradient?



(a) The line segments with a positive gradient are: AB, GH, and KL.

(b) The line segments with a negative gradient are: CD, EF and IJ.

$y = mx + c$

This is the **equation of a straight line** or **linear function**. x is an x-coordinate and we can use the equation to calculate the corresponding y-coordinate, y, if we know the values of m and c. Examples of linear functions include:

$y = 3x + 4$	$y = 2x + 7$
$y = 3x$ (as c may be 0)	$y = 4x - 5$ (as c may be minus)
$y = 6 - 6x$ (x may be minus)	$y = x$

$y = ½x - 8$ (fractions are possible)

$2y = x - 16$ (the same thing as $y = ½x - 8$!)

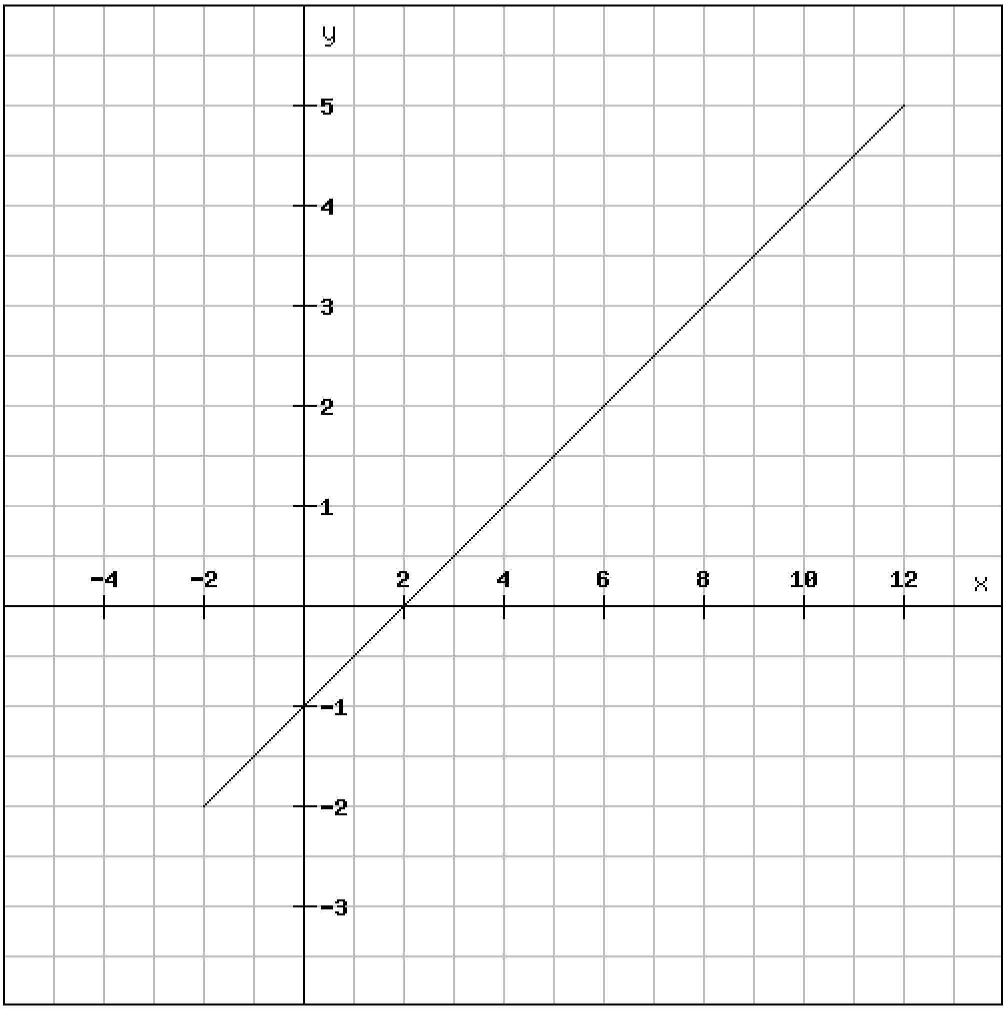
$23y = 7x - 13$	$y = \frac{7x-13}{23}$ (same as previous)

and so on

Plotting Points in All Four Quadrants

Some graphs are only concerned with positive values, especially those that use real-life data. For instance, we might want to plot the height and weight of everybody in the class. We know that no one will have a negative height or weight.

But often you will need to take negative values into account. For instance you might be asked to plot the graph of $y = ½x - 1$ for all values of y from –2 to +5. Your graph (drawn accurately on graph paper!) should look something like this:



This graph has four quadrants. Above the *x*-axis and to the right of the *y*-axis (the axes are the thicker lines, often arrowed upwards and right), we have the quadrant in which both *x* and *y* are positive. The bottom left quadrant, on the other hand, is for negative values of both *x* and *y*. A straight line may also pass through a third quadrant (here, the one where *x* is positive but *y* is negative).

We can think of the equation $y = ½x - 1$ as a **function**, that is a rule for generating *y* values from *x* values. Put *x* values into the function and get *y* values out.

Here is the table of values that needed plotting:

x	y
–2	$½(-2) - 1 = -2$
0	$½(0) - 1 = -1$
2	$½(2) - 1 = 0$
4	$½(4) - 1 = 1$

6	$\frac{1}{2}(6) - 1 = 2$
8	$\frac{1}{2}(8) - 1 = 3$
10	$\frac{1}{2}(10) - 1 = 4$
12	$\frac{1}{2}(12) - 1 = 5$

This gives us the co-ordinates that should be located on the graph. With the x-coordinate given first, these are (–2, –2) through to (12, 5). If these two are plotted correctly, we can be confident that a straight line will pass through all the other points! But, it is good practice to plot *three* points to be sure that a straight line is correct.

Please note that here we have only drawn a **section** of the line $y = \frac{1}{2}x - 1$. This was the section for values of y from –2 to +5. Could we draw *all* of the line of $y = \frac{1}{2}x - 1$? No, this line continues indefinitely in both directions. So you will normally be asked to plot a specific section. Make sure you follow those instructions carefully.

The Gradients of Parallel Lines

If the graph of $y = \frac{1}{2}x - 1$ can be plotted as a straight line in this way, what would the graph of $y = \frac{1}{2}x - 6$ or $y = \frac{1}{2}x + 4$ look like?

Complete this table of values for the finctions $y = \frac{1}{2}x - 6$ and $y = \frac{1}{2}x + 4$ and plot them on the grid with $y = \frac{1}{2}x - 1$.

x	$y = \frac{1}{2}x - 6$	$y = \frac{1}{2}x + 4$
–2	$\frac{1}{2}(-2) - 6 = -7$	$\frac{1}{2}(-2) + 4 = 3$
0	$\frac{1}{2}(0) - 6 = -6$	
2	$\frac{1}{2}(2) - 6 =$	
4		
6		
8		
10		
12		

Answer: these will be parallel lines, shifted to left or right on your graph. Try it and see! Parallel lines are said to have the same gradient (slope). As long as $y = \frac{1}{2}x$ remains in our equation, the gradient will remain the same. But as soon as we change it to $y = 2x - 1$ (say), we get a straight line with a different gradient.

We can say that the statement $y = ax \pm k$ (where **a** is a specified integer and **k** is any constant) *defines* the gradient.

We now study this equation in detail. The letters m and c have important meanings.

The letter c represents the 'intercept' on the y-axis. This just means the y co-ordinate of the point where the line crosses the y-axis.

The letter m represents the gradient of the straight line.

Example 3

Write down the gradient and the intercept on the y axis for each of the following equations:

(a) $y = 2x + 5$
(b) $y = x + 3$
(c) $y = -4x + 0.2$
(d) $y = 3x - 1$
(e) $y = 4 - 10x$
(f) $y = \frac{1}{2}x - \frac{3}{4}$

	Equation	**Gradient**	**Intercept**	**Comment**
(a)	$y = 2x + 5$	2	5	
(b)	$y = x + 3$	1	3	Think of x as $1 \times x$
(c)	$y = -4x + 0.2$	-4	0.2	
(d)	$y = 3x - 1$	3	-1	
(e)	$y = 4 - 10x$	-10	4	4 – 10x is the same as $-10x + 4$
(f)	$y = \frac{1}{2}x - \frac{3}{4}$	$\frac{1}{2}$	$-\frac{3}{4}$	

Yes, but what do 'gradient' and 'intercept' actually look like?

Consider the three straight lines:

$y = x$

$y = x + 3$

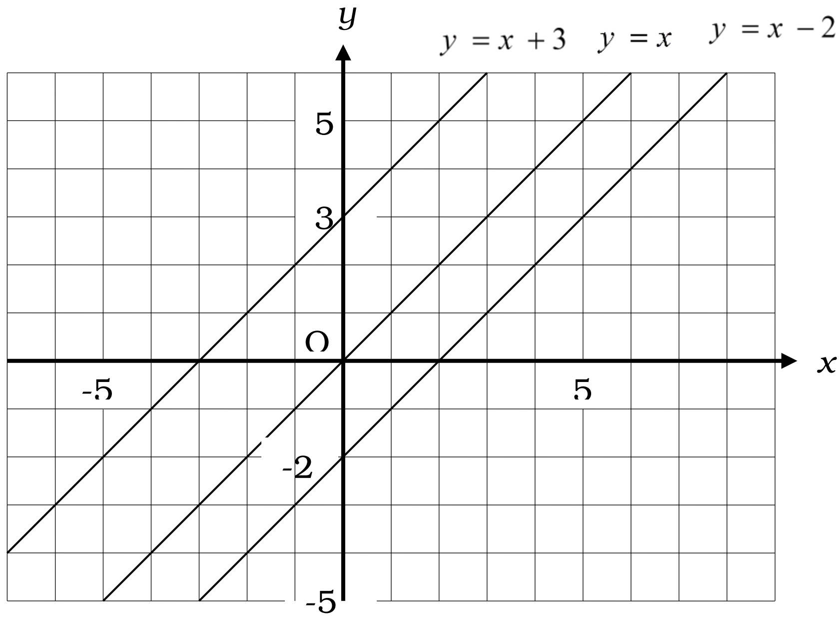
$y = x - 2$

The diagram below shows these three equations plotted on a grid.

All three lines have the same gradient, which is 1 (remember that we can think of x as $1 \times x$). But gradient is a measure of the slope of a line – how steep it is. If all three lines have the same slope, then they are parallel to each other.

Parallel lines have the same gradient, *m*.

Now look where each line meets the y axis. The first line meets the y axis at the origin (0, 0). This makes sense if we think of the equation $y = x$ as $y = x + 0$. The second line meets the y axis at the point (0, 3), which makes sense: c is 3 in this equation. The third line meets the y axis at the point (0, –2), which also makes sense: c is –2 in this equation.



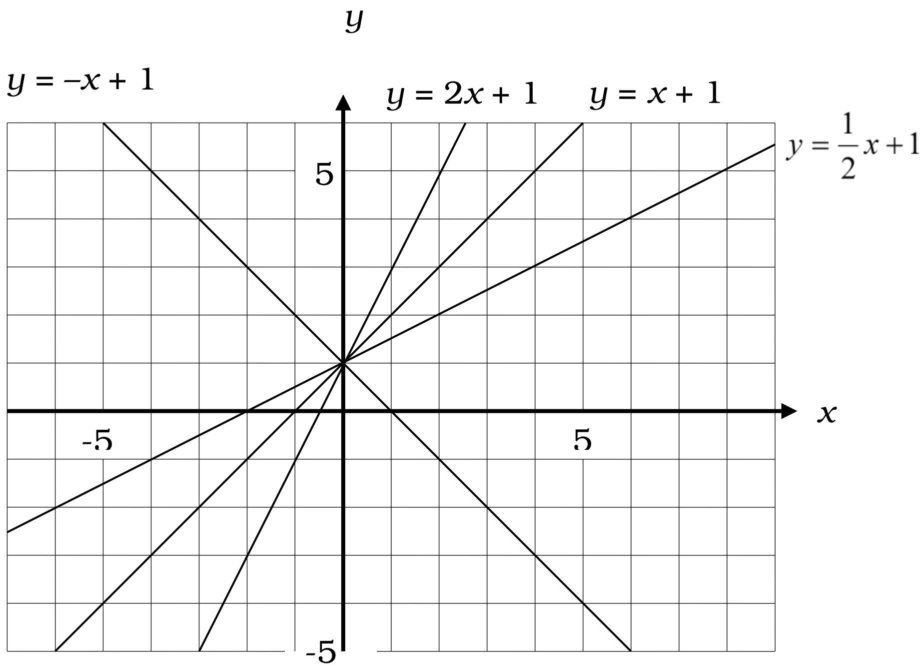
Now consider another set of four equations:

$$y = x + 1$$
$$y = 2x + 1$$
$$y = \frac{1}{2}x + 1$$
$$y = -x + 1$$

These four equations are plotted on the grid below. Note that all four equations pass through the point (0, 1). This makes sense, since c, the 'intercept' on the y axis, has the value one for all four equations.

The first three equations are 'uphill': they have a positive gradient. This is to be expected, since the values of m are positive for these first three equations (1, 2 and ½ respectively). The fourth equation, however, is 'downhill': it has a negative gradient. This is also to be expected since the value of m is -1 for this equation.

Look closely at lines corresponding to the first three equations. The steepest line, $y = 2x + 1$, has the largest gradient (2), while the least steep line, $y = \frac{1}{2}x + 1$, has the smallest gradient (1/2).

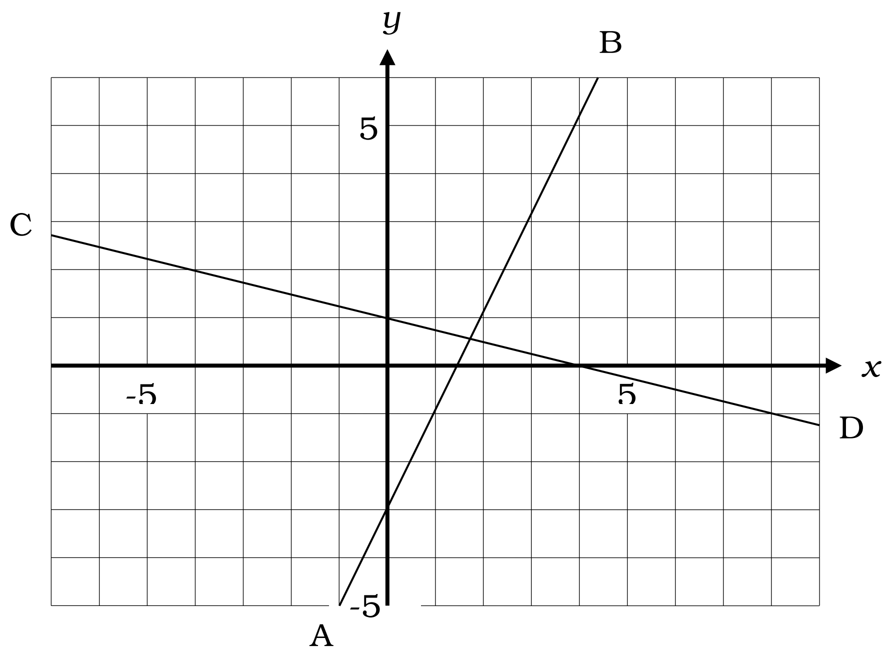


Graph to Equation?

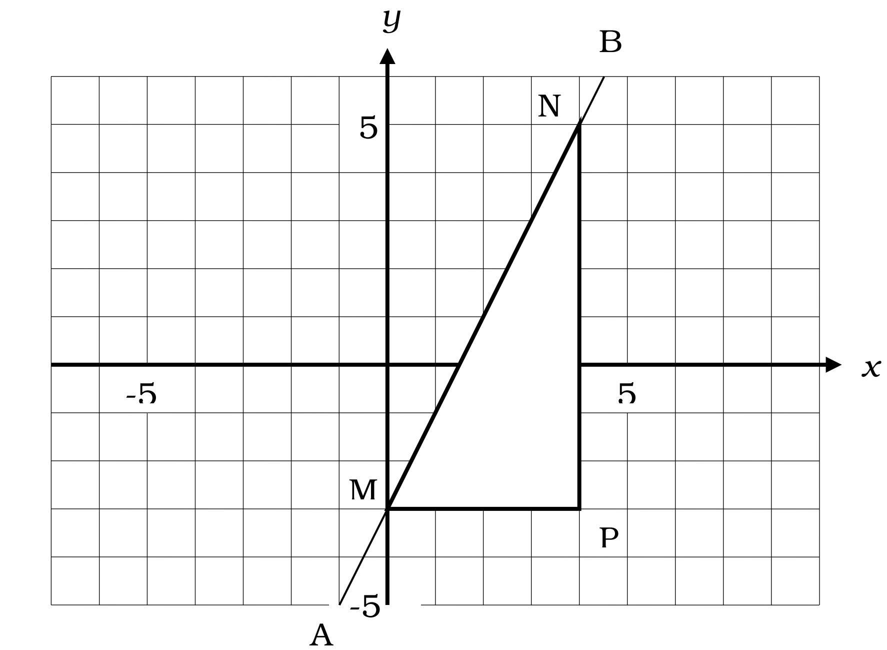
The aim is to identify the values of c and m from the graph, and then write the required equation in the form $y = mx + c$.

Example 4

Find the equations of the straight lines AB and CD.

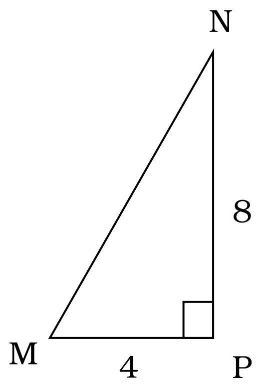


The line AB passes through the point (0, –3). The value of c is therefore –3. The value of m requires a little more work. We use the original definition of gradient as $\frac{change\ in\ y}{change\ in\ x}$. Make any right-angled triangle with a part of AB as hypotenuse. Any such triangle will do, but a larger triangle will often be more accurate. However, the most important consideration is the convenience of the arithmetic.



The right-angle triangle MNP is just one suitable choice. We see from the grid that NP is 8 units and that MP is 4 units. The gradient of MN, and therefore of AB is:

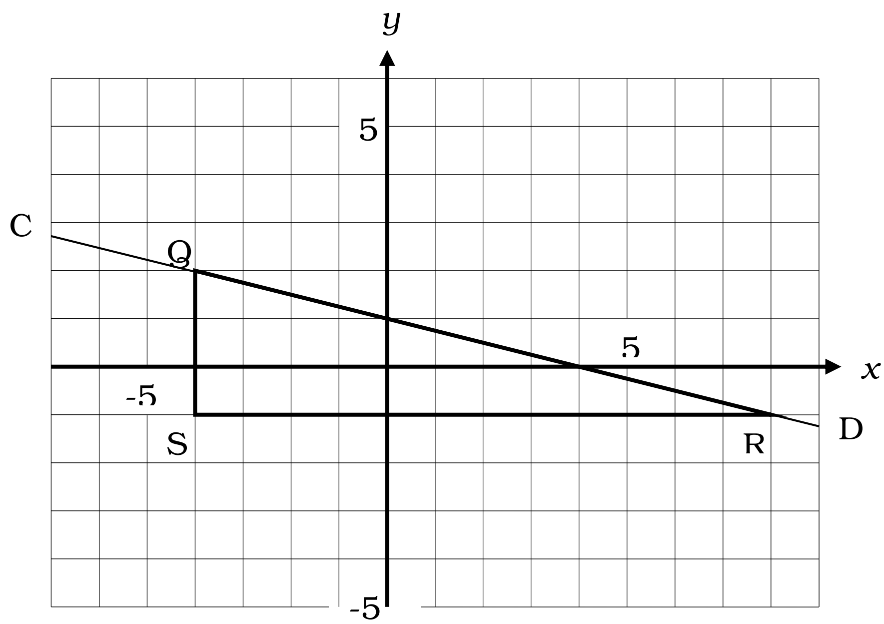
$$\frac{\text{change in } y}{\text{change in } x} = \frac{NP}{MP} = \frac{8}{4} = 2$$



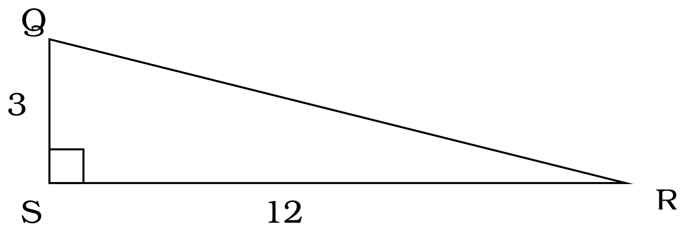
The value of m is 2, and the equation of the straight line AB is therefore $y = 2x - 3$.

The line CD crosses the y-axis at the point (0, 1). The value of c is therefore 1.

To find the gradient of the line CD, draw any right-angle triangle that has part of CD as hypotenuse. One convenient choice might be triangle QRS.



We see from the grid that the lengths of QS and SR are 3 and 12 units respectively. However, as we move left to right, in the direction of increasing *x*, the value of *y* **decreases**, since QR is 'downhill'.

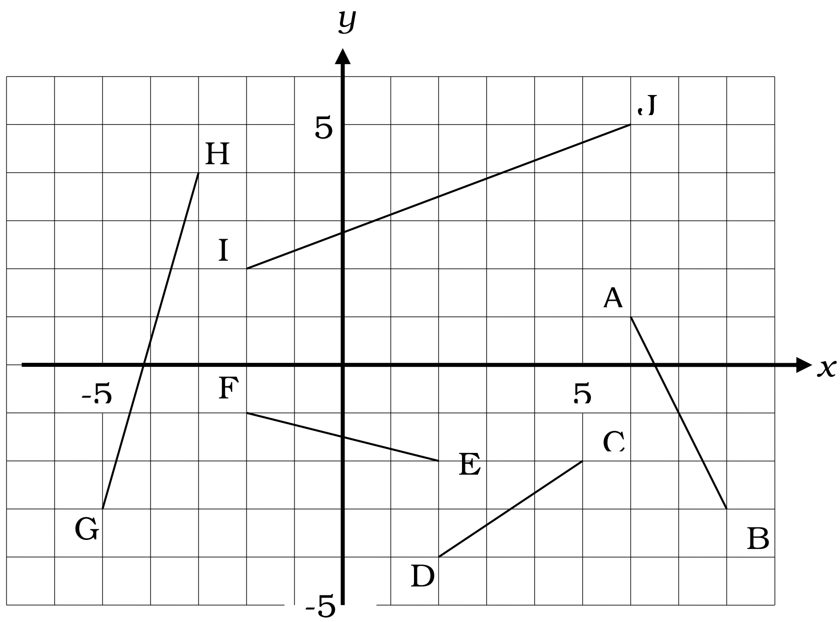


The gradient of QR is $\frac{\text{change in } y}{\text{change in } x} = \frac{-3}{12} = -\frac{1}{4}$

The equation of the straight line CD is therefore $y = -\frac{1}{4}x + 1$.

Activity 1

1 Which of the following line segments has:
(a) a positive gradient
(b) a negative gradient?

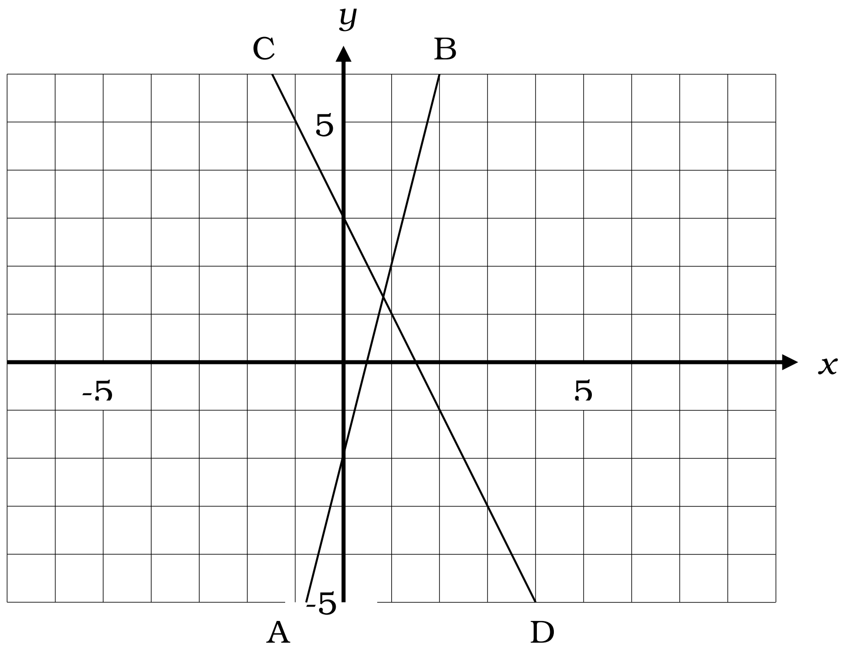


2 Find the gradient of the line segment AB where A and B have co-ordinates:
(a) A(3, 10), B(7, 30)
(b) A(18, 9), B(6, 6)
(c) A(-2, –5), B(3, 10)
(d) A(10, 15), B(-4, 8)
(e) A(5, 12), B(8, 6)
(f) A(-10, –2), B(-8, –10)

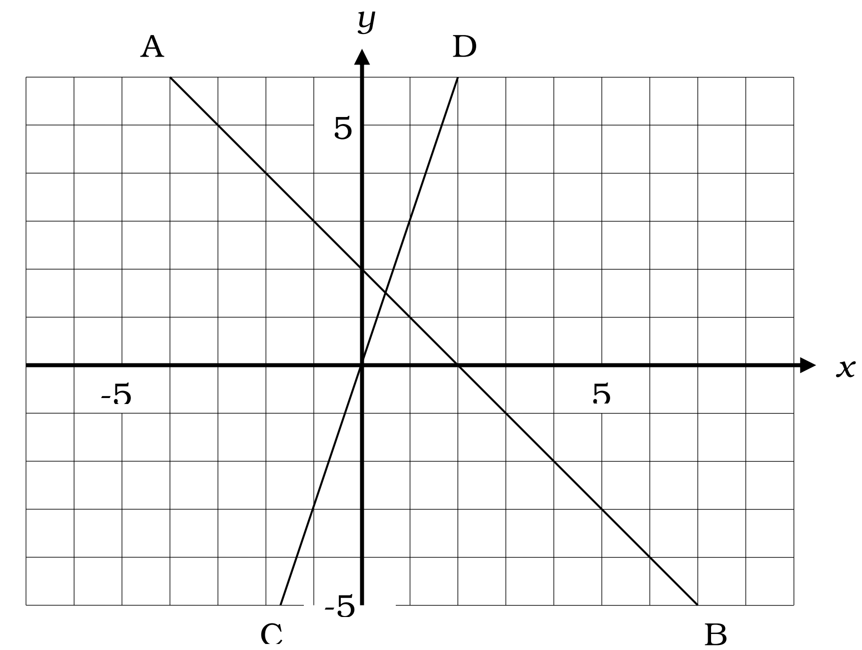
3 Write down the gradient and intercept on the *y* axis for the following equations:
(a) $y = 4x - 2$
(b) $y = \frac{1}{3}x + 7$
(c) $y = x - \frac{4}{5}$
(d) $y = 0.6 - x$

4 Identify the parallel lines:
$L_1 = 2x - 7$; $L_2 = -2x - 7$; $L_3 = 2x - 5$; $L_4 = x - 7$; $L_5 = -2x + 7$

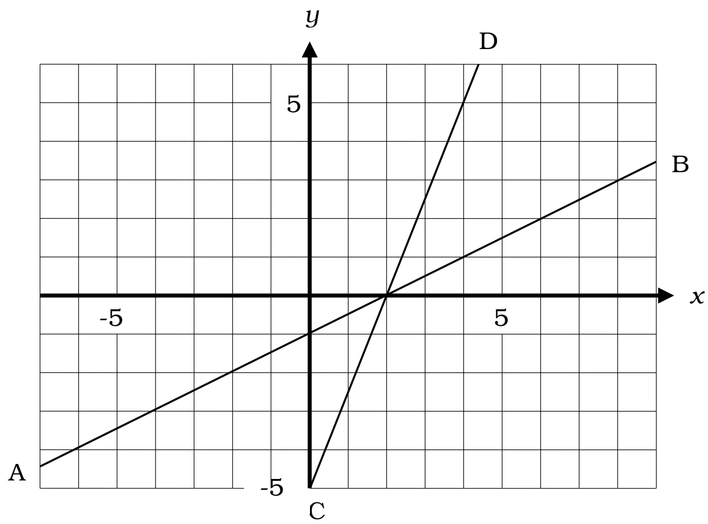
5 Find the equations of the two lines AB and CD:



6 Find the equations of the two lines AB and CD:



7 Find the equations of the two lines AB and CD:



Writing the equation of a line without a graph

You may be given the gradient of a line and a point on a line and asked to find the equation of the line.

Example 1

The gradient of a line is 2. Given that the point (2,7) lies on the line find the equation of the line in the form $y = mx + c$.

We can put the gradient information straight into $y = mx + c$

$$y = 2x + c$$

Now we need to find the value of c. We do this using the point that we know is on the line. Substitute the values of x and y from the given coordinates being careful to get them the right way round!

$$7 = 2 \times 2 + c$$

$$7 = 4 + c$$

From this we can see that $c = 3$.

Using our value of c, the required equation is: $y = 2x + 3$

Example 2

You may be be given two points and asked to find the equation of the line that they both lie on or that joins them.

Find the equation of the line joining (-1,-5) and (1,3)

The first thing we need is a gradient. Find this by comparing the y coordinates and the x coordinates since we know that

$$\text{Gradient} = \frac{\text{change in } y}{\text{change in } x}$$

The y coordinate changes from –5 to 3, a change of 8. The x coordinate changes from –1 to 1, a change of 2.

$$gradient = \frac{8}{2} = 4$$

Now we can put the gradient information into $y = mx + c$

$$y = 4x + c$$

and use the coordinates of one of the points to find c as above. We can choose either of the points we have been given and should get the same answer. (1, 3) is the more straightforward choice in this case since the numbers are positive.

$$3 = 4 \times 1 + c$$

From this we can see that $c = –1$ and the required equation is

$$y = 4x - 1$$

Activity 2	Find the equations of the straight lines in the form $y = mx + c$

1. (a) gradient =3, passing through the point (1,5)
   (b) gradient =1, passing through the point (–1,–4)
   (c) gradient =1/2, passing through the point (2,2)
   (d) gradient =–2, passing through the point (0,5)

2. (a) passing through the points (1,–6) and (–1,16)
   (b) passing through the points (0,2) and (–1,1)
   (c) passing through the points (–1,–1) and –2,1)
   (d) passing through the points (3,2) and 6,3)

Using and interpreting gradients and linear functions

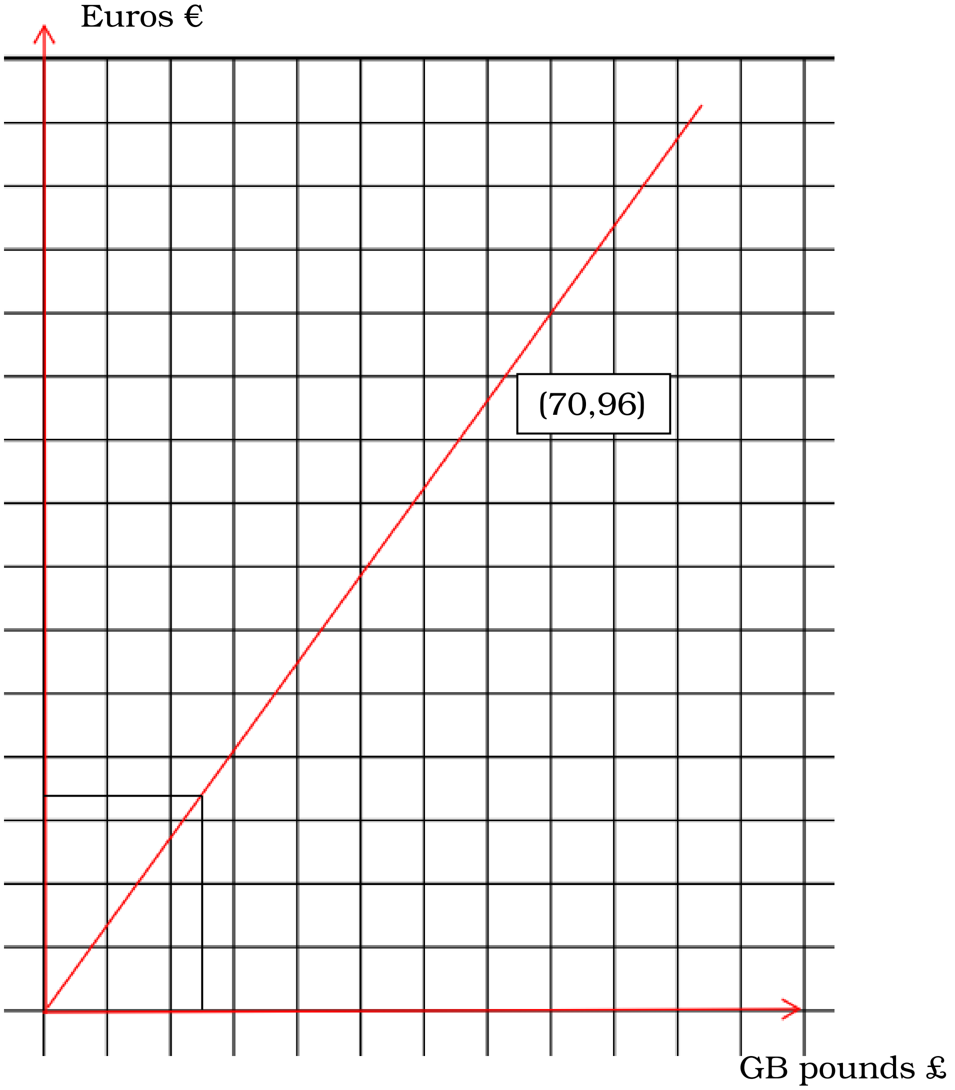
Linear functions which pass through the origin (i.e. $c = 0$) describe **direct proportion**. This is an idea we will come back to later when we look at formulae. When two quantities are in direct proportion they increase at the same rate, if you double one, the other doubles as well. The example below is an example of direct proportion.

Example

The graph below shows the exchange rate for changing £ into Euro's.

(a) Use the graph to estimate the number of Euros you would get in exchange for £25.

(b) Find the gradient of the graph and explain its meaning.



(a) From the graph we can see that £25 is approximately equivalent to €33.

(b) Comparing the given point (70,96) and (0,0)

the change in y from (0,0) to (70, 96) is 96 and
the change in x from (0,0) to (70, 96) is

gradient = $\frac{\text{change in } y}{\text{change in } x} = \frac{96}{70} = 1.37$

This is the exchange rate, it is the number of Euros per pound.

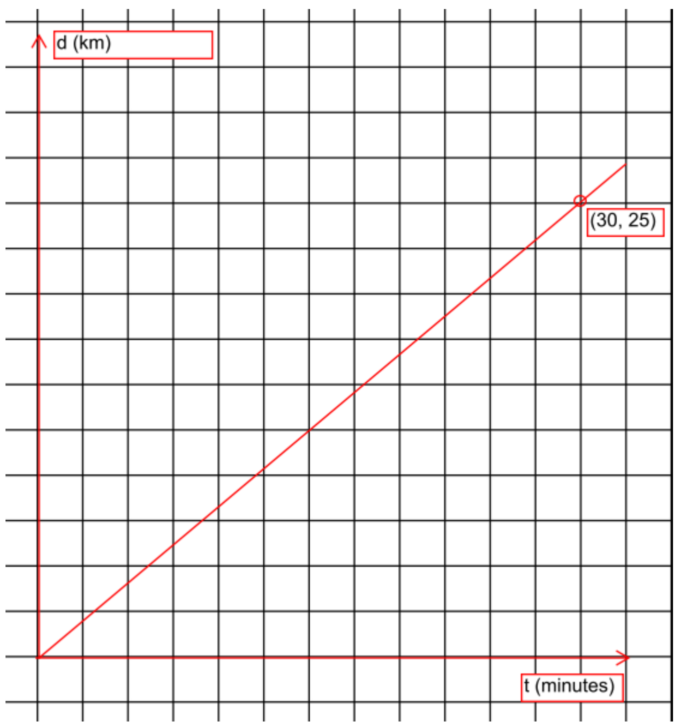
We could write down the rule $e = 1.37 \times p$ where e is the amount in € and p is the amount in £. This is in the same format as $y = mx + c$. Instead of y we have the amount in euros, e, and instead of x we have the amount in £, p. The gradient, m, is 1.37 and the y-intercept, c, is 0 since the line goes through (0,0).

Activity 3

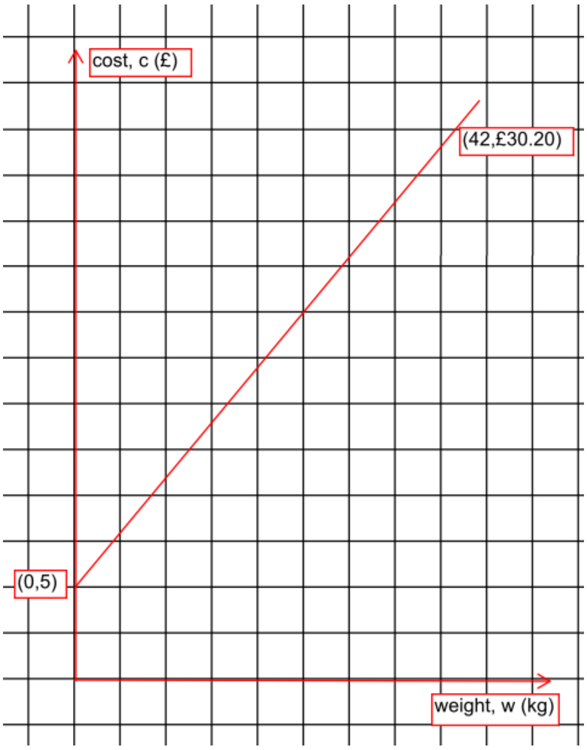
In each of these questions find the gradient of line segment and describe its meaning.

Challenge: Write down a rule relating the two variables in the form of $y = mx + c$ (though your variables will be not be x and y!)

1. A cyclist rides for 30 minutes and covers a distance of 25km, as shown in the diagram. Find the gradient of the line and describe its meaning.



2. The cost of sending a parcel through a courier service is shown in the diagram. Find the gradient of the line and describe its meaning.



Suggested Answers to Activities

Activity One

1. (a) CD, GH, IJ
   (b) AB, EF

2. (a) 5
   (b) ¼
   (c) 3
   (d) ½
   (e) –2
   (f) –4

3.

	Gradient	Intercept on *y* axis
(a)	4	(0, –2)
(b)	1/3	(0, 7)
(c)	1	(0, –4/5)
(d)	-1	(0. 0.6)

4. L_1 is parallel to L_3 and L_2 is parallel to L_5

5. AB: $y = 4x - 2$
   CD: $y = -2x + 3$ or $y = 3 - 2x$

6. AB: $y = -x + 2$ or $y = 2 - x$
   CD: $y = 3x$

7. AB: $y = \frac{1}{2}x - 1$ or $y = \frac{x}{2} - 1$
   CD: $y = \frac{5}{2}x - 5$ or $y = 2.5x - 5$

Activity Two

1. (a) $y = 3x + 2$; (b) $y = x - 3$
   (c) $y = \frac{1}{2}x + 1$; (d) $y = -2x + 5$

2. (a) $y = 5 - 11x$; (b) $y = x + 2$
   (c) $y = -2x - 3$; (d) $y = \frac{1}{3}x + 1$

Activity Three

1. Comparing the points (0,0) and (30,25) the $gradient = \frac{25-0}{30-0} = \frac{25}{30} = \frac{5}{6} km/\min$ This is the speed of the cyclist, measured in km per minute.
   The rule is $d = \frac{5}{6}t$
2. Comparing the points (0,£5) and (42,£30.20) the

   $gradient = \frac{30.2-5}{42-0} = \frac{25.20}{42} = 0.6$ *cost/kg*. The cost of sending a parcel is calculated using 0.6 × *weight* plus a fixed charge of £5 for every parcel. The gradient is the cost per kg.
   The rule is *c = 0.6 × w + 5.*